Online Appendix for "Extrapolation and Bubbles," by Nicholas Barberis, Robin Greenwood, Lawrence Jin, and Andrei Shleifer

In this Appendix, we show that the most important predictions of our model continue to hold even when we replace the boundedly-rational fundamental traders by fully rational traders. Specifically, we show that a sequence of strongly positive cash-flow shocks again generates a large overvaluation relative to fundamental value, and also that a significant fraction of the volume during the height of the bubble consists of trading among the wavering extrapolators.

In the model we study below, all investors can short. We have removed the short-sale constraint for tractability, but also to demonstrate that, as claimed in Section 4, our main conclusions do not depend on the short-sale constraint, but only on the presence of wavering extrapolators.

Model setup

Asset structure

There is a risky asset which pays a single dividend at time T that evolves as:

$$D_T = D_0 + \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_T, \qquad \varepsilon_t \ i.i.d. \sim \mathcal{N}(0, \sigma_\varepsilon^2), \forall t. \tag{1}$$

Investor wealth not invested in the risky asset can be carried into the future at an interest rate that is normalized to zero.

The price of the risky asset at time t is denoted P_t, and its per-capita supply is fixed at Q.

Investors

There are two types of investors in the economy: extrapolators and fully rational traders, who constitute a fraction μ_E and μ_R of the total population, respectively, so that

$$\mu_E + \mu_R = 1. \tag{2}$$

As in Propositions 2 and 3 of the paper, there is a continuum of extrapolators, and each extrapolator draws an independent weight $w_{i,t}$ at time t from a bounded and continuous density $g(w)$, $w \in [w_l, w_h]$, with mean $\overline{w}$ and with $0 < w_l < w_h < 1$. These weights are independent across extrapolators and independent over time, and

$$\int_{w_l}^{w_h} g(w)dw = \mu_E. \tag{3}$$

Type i extrapolators' demand function at time t is

$$N_t^E(w_{i,t}) = w_{i,t} \frac{D_t - \gamma\sigma_\varepsilon^2 (T-t-1)Q - P_t}{\gamma\sigma_\varepsilon^2} + (1 - w_{i,t}) \frac{X_t}{\gamma\sigma_\varepsilon^2}, \qquad t = 0, 1, \ldots, T-1, \tag{4}$$

where

$$X_t = \begin{cases} \gamma\sigma_\varepsilon^2 Q & t=0 \\ (1-\theta)\sum_{k=1}^{t-1}\theta^{k-1}(P_{t-k}-P_{t-k-1})+\theta^{t-1}X_1 & t\geq 1 \end{cases}. \quad (5)$$

Note that, for any $t \geq 1$, we have the following recursive relation for X_t:

$$X_{t+1} = \theta X_t + (1-\theta)(P_t - P_{t-1}). \quad (6)$$

The second type of investors are rational traders who are fully aware of extrapolators' demand in (4), the structure of extrapolator wavering, and the market clearing condition

$$\int_{w_l}^{w_h} g(w)N_t^E(w)dw + \mu_R N_t^R = Q, \quad t=0,1,2,\ldots,T-1. \quad (7)$$

To determine their demand N_t^R, the rational investors take the current price as given and solve the following problem

$$\max_{N_t^R} \mathbb{E}_t^R\left\{-\exp\left(-\gamma[W_t^R + N_t^R(P_{t+1}-P_t)]\right)\right\}. \quad (8)$$

Proposition. *In this economy, the price of the risky asset at time t is given by*

$$P_t = f_{T-t}^{-1}\left\{g_{T-t}D_t - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[\begin{array}{l} \frac{\mu_E(1-\overline{w})(1-\theta)}{\gamma\sigma_\varepsilon^2}P_{t-1}\sum_{n=0}^{T-t-2} f_n^{-1}g_n^2\theta^{T-t-2-n} + Q\sum_{l=0}^{T-t-1} f_l^{-1}g_l^2 \\ +\mu_E\overline{w}Q\sum_{s=0}^{T-t-1} sf_s^{-1}g_s^2 - \frac{\mu_E(1-\overline{w})}{\gamma\sigma_\varepsilon^2}X_t\sum_{n=0}^{T-t-1} f_n^{-1}g_n^2\theta^{T-t-1-n} \end{array}\right]\right\}, \quad (9)$$

where the two factors, f and g, are defined by

$$\begin{aligned} f_n &= f_{n-1} + \tfrac{\mu_E\overline{w}}{\mu_R} f_{n-1}^{-1}g_{n-1}^2 + \tfrac{\mu_E(1-\overline{w})}{\mu_R}(1-\theta)\sum_{l=0}^{n-3} f_l^{-1}g_l^2\theta^{n-3-l} - \tfrac{\mu_E(1-\overline{w})}{\mu_R}(1-\theta)\sum_{m=0}^{n-2} f_m^{-1}g_m^2\theta^{n-2-m}, \\ g_n &= g_{n-1} + \tfrac{\mu_E\overline{w}}{\mu_R} f_{n-1}^{-1}g_{n-1}^2, \end{aligned} \quad (10)$$

for $n \geq 1$, and $f_0 = g_0 = 1$.

Proof of Proposition. We derive the equilibrium price of the risky asset through the maximization problem of the fully rational investors. We assume and later verify that, conditional on all information up to time *t*, the rational (i.e. true) distribution of P_{t+1} is Normal. Given this assumption, (8) leads to

$$P_t = \mathbb{E}_t^R(P_{t+1}) - \tfrac{\gamma}{\mu_R}\mathrm{Var}_t^R(P_{t+1}-P_t)\left[Q - \int_{w_l}^{w_h} g(w)N_t^E(w)dw\right]. \quad (11)$$

We now apply backward induction to derive the equilibrium risky asset price. At $t = T-1$, $D_{T-1}, w_{i,T-1}, \forall i,$ and P_{T-1} are all realized or known. Knowing that $P_T = D_T$, we must have

$$\mathbb{E}_{T-1}^R(P_T) = D_{T-1}, \qquad \mathrm{Var}_{T-1}^R(P_T - P_{T-1}) = \sigma_\varepsilon^2. \quad (12)$$

As a result

$$
\begin{aligned}
P_{T-1} &= D_{T-1} - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \frac{D_{T-1}-P_{T-1}}{\gamma\sigma_\varepsilon^2}\int_{w_l}^{w_h} g(w)w\,dw - \frac{X_{T-1}}{\gamma\sigma_\varepsilon^2}\int_{w_l}^{w_h} g(w)(1-w)\,dw\right] \\
&= D_{T-1} - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \frac{D_{T-1}-P_{T-1}}{\gamma\sigma_\varepsilon^2}\mu_E\overline{w} - \frac{X_{T-1}}{\gamma\sigma_\varepsilon^2}\mu_E(1-\overline{w})\right].
\end{aligned} \tag{13}
$$

Rearranging terms gives

$$
\begin{aligned}
P_{T-1} &= \left(1+\mu_R^{-1}\mu_E\overline{w}\right)^{-1} \\
&\quad \left\{\left(1+\mu_R^{-1}\mu_E\overline{w}\right)D_{T-1} - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \frac{X_{T-1}}{\gamma\sigma_\varepsilon^2}\mu_E(1-\overline{w})\right]\right\}.
\end{aligned} \tag{14}
$$

At $t = T-2$, (14) implies that P_{T-1} has a Normal distribution. As a result, it is valid to use (11) to determine the risky asset price at $t = T-2$, and the rational investors compute the conditional expectation and conditional variance of P_{T-1} as follows

$$
\begin{aligned}
\mathbb{E}_{T-2}^R(P_{T-1}) &= \left(1+\mu_R^{-1}\mu_E\overline{w}\right)^{-1} \\
&\quad \left\{\left(1+\mu_R^{-1}\mu_E\overline{w}\right)D_{T-2} - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \frac{\mu_E(1-\overline{w})}{\gamma\sigma_\varepsilon^2}[\theta X_{T-2}+(1-\theta)(P_{T-2}-P_{T-3})]\right]\right\}.
\end{aligned} \tag{15}
$$

$$
\mathrm{Var}_{T-2}^R(P_{T-1}-P_{T-2}) = \sigma_\varepsilon^2. \tag{16}
$$

Substituting (15) and (16) back into (11) for $t = T-2$ gives

$$
\begin{aligned}
P_{T-2} &= \left(1+\mu_R^{-1}\mu_E\overline{w}\right)^{-1} \\
&\quad \left\{\left(1+\mu_R^{-1}\mu_E\overline{w}\right)D_{T-2} - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \frac{\mu_E(1-\overline{w})}{\gamma\sigma_\varepsilon^2}[\theta X_{T-2}+(1-\theta)(P_{T-2}-P_{T-3})]\right]\right\} \\
&\quad - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \int_{w_l}^{w_h} g(w)N_{T-2}^E(w)\,dw\right] \\
&= \left(1+\mu_R^{-1}\mu_E\overline{w}\right)^{-1} \\
&\quad \left\{\left(1+\mu_R^{-1}\mu_E\overline{w}\right)D_{T-2} - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \frac{\mu_E(1-\overline{w})}{\gamma\sigma_\varepsilon^2}[\theta X_{T-2}+(1-\theta)(P_{T-2}-P_{T-3})]\right]\right\} \\
&\quad - \frac{\gamma\sigma_\varepsilon^2}{\mu_R}\left[Q - \mu_E\overline{w}\frac{D_{T-2}-\gamma\sigma_\varepsilon^2 Q - P_{T-2}}{\gamma\sigma_\varepsilon^2} - \mu_E(1-\overline{w})\frac{X_{T-2}}{\gamma\sigma_\varepsilon^2}\right].
\end{aligned} \tag{17}
$$

Rearranging terms gives

$$P_{T-2} = f_2^{-1}\Big\{g_2 D_{T-2} - \tfrac{\gamma\sigma_\varepsilon^2}{\mu_R}\Big[\tfrac{\mu_E(1-\overline{w})(1-\theta)}{\gamma\sigma_\varepsilon^2} P_{T-3} + \Big(1+\Big(1+\tfrac{\mu_E\overline{w}}{\mu_R}\Big)\Big)Q$$
$$+\Big(1+\tfrac{\mu_E\overline{w}}{\mu_R}\Big)\mu_E\overline{w}Q - \tfrac{\mu_E(1-\overline{w})}{\gamma\sigma_\varepsilon^2}\Big(1+\tfrac{\mu_E\overline{w}}{\mu_R}+\theta\Big)X_{T-2}\Big]\Big\}. \tag{18}$$

where

$$f_2 = 1+\tfrac{\mu_E\overline{w}}{\mu_R}+\Big(1+\tfrac{\mu_E\overline{w}}{\mu_R}\Big)\tfrac{\mu_E\overline{w}}{\mu_R}-\tfrac{\mu_E(1-\overline{w})(1-\theta)}{\mu_R},$$
$$g_2 = \Big(1+\tfrac{\mu_E\overline{w}}{\mu_R}\Big)+\Big(1+\tfrac{\mu_E\overline{w}}{\mu_R}\Big)\tfrac{\mu_E\overline{w}}{\mu_R}. \tag{19}$$

Continuing in this way leads to expressions (9) and (10). In addition, (9) and (10) verify the assumption that, conditional on all information up to time *t*, the rational distribution of P_{t+1} is indeed Normal. ■

Asset pricing implications

We first examine the pricing implications of this model. Specifically, we study an economy where 70% of investors are extrapolators and 30% are fully rational traders, and consider the same sequence of 50 cash-flow shocks that we used in several of the examples in the paper: 10 shocks of zero, followed by shocks of 2, 4, 6, 6, followed by 36 shocks of zero. In Figure A1, we plot the price of the risky asset and the fundamental value of the asset.

Figure A1 can be directly compared to Figure 1 in the paper. All else equal, removing the short-sale constraint for all investors tends to reduce the size of the bubble: in an economy where 70% of investors are extrapolators and 30% are boundedly-rational fundamental traders and where all investors can short, the bubble size is typically small. However, replacing the boundedly-rational fundamental traders by fully rational traders tends to *increase* the size of the bubble: since rational traders are fully aware of the persistence of extrapolator beliefs, they do not trade aggressively against mispricing. Interestingly, Figure A1 shows that the second effect can dominate: the bubble in this example is *larger* than the bubble presented in Figure 1 of the paper. A sequence of strongly positive cash-flow shocks can therefore lead to a large overvaluation even when fully rational traders are present in the economy, and even when all investors can short.

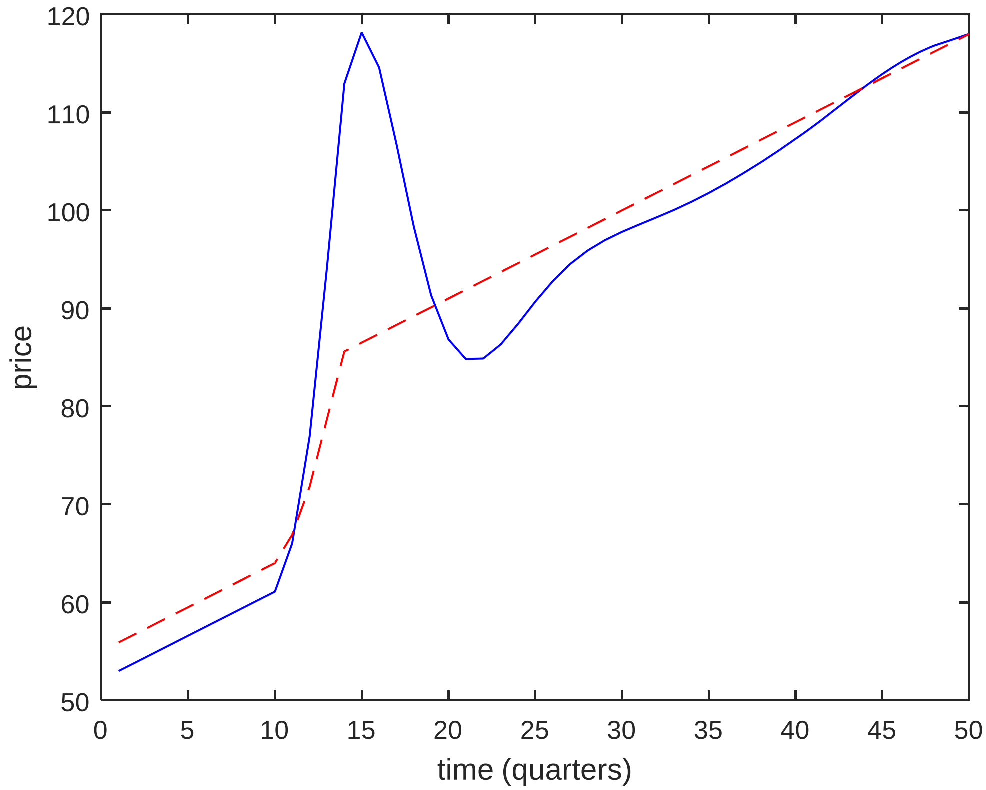

Figure A1: Prices in a bubble. The solid line plots the price of the risky asset for the following sequence of 50 cash-flow shocks: 10 shocks of zero, followed by shocks of 2, 4, 6, 6, followed by 36 shocks of zero. 30% of the investors are *rational* traders; the remaining 70% are a continuum of extrapolators with an extrapolation parameter θ of 0.9 and where each extrapolator draws a weight w on the value signal from a bounded and continuous density $g(w)$ with mean $\overline{w}$ of 0.1. The dashed line plots the fundamental value of the asset for the same cash-flow sequence. The other parameters are $D_0 = 100$, $\sigma_\varepsilon = 3$, $Q = 1$, and $\gamma = 0.1$.

Volume implications

We now look at the trading implications of the model with fully rational traders. While we would like to compute trading volume for a *finite* number of extrapolator types, the pricing equation in (9) assumes a continuum of extrapolators. To proceed, we assume that the equilibrium price follows (9) exactly even when there are I types of extrapolators rather than a continuum of them; we have checked that this is a very accurate approximation.

In Figure A2, we plot the total trading volume in the risky asset (solid line) and the trading volume between extrapolators (dashed line) in an economy where 70% of investors consist of 50 extrapolator types and 30% are fully rational traders. The cash-flow shocks are the same as those in Figure A1. For the 50 types of extrapolators, their weights on the value signal are generated by the wavering model in equation (8) of the paper. Also as in the paper, we truncate the wavering component $\tilde{u}_{i,t}$ at $\pm 0.9\min(1-\overline{w}, \overline{w})$.

Figure A2 below can be directly compared to Figure 4 in the paper. Removing the short-sale constraint and replacing fundamental traders by fully rational traders increases the total trading volume at the peak of the bubble because shorting allows rational investors to trade heavily during bubbles. At the same time, trading *between* extrapolators makes up a significant portion (around 40%) of total trading volume. In other words, the prediction of our original model that a significant amount of trading during bubbles comes from trading *between* extrapolators continues to hold even in the presence of fully rational traders.

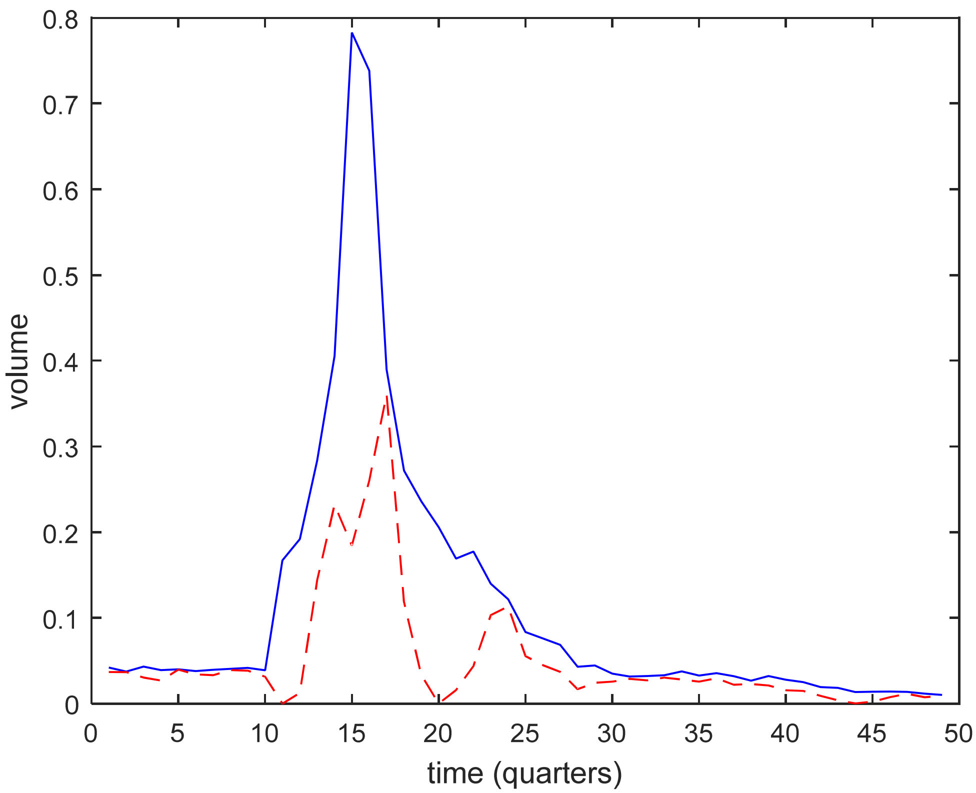

Figure A2: Volume in a bubble. The solid line plots the total trading volume in the risky asset for the following sequence of 50 cash-flow shocks: 10 shocks of zero, followed by shocks of 2, 4, 6, 6, followed by 36 shocks of zero. The dashed line plots the trading volume between the extrapolators for the same cash-flow sequence. 30% of the investors are rational traders; the remaining 70% are 50 types of extrapolators with an extrapolation parameter θ of 0.9 and where each extrapolator puts a base weight $\overline{w}$ of 0.1 on the value signal. The other parameters are $D_0 = 100$, $\sigma_\varepsilon = 3$, $Q = 1$, $\gamma = 0.1$, and $\sigma_u = 0.03$.