A Speculative Asset Pricing Model of Financial Instability[*]

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ABSTRACT

I develop a dynamic equilibrium model that incorporates incorrect beliefs about crash risk and use it to explain the available empirical evidence on financial booms and busts. In the model, if a long period of time goes by without a crash, some investors' perceived crash risk falls below the true crash risk, inducing them to take on excessive leverage. Following a drop in fundamentals, these investors de-lever substantially, both because of their high pre-crash leverage and because they now believe future crashes to be more likely. Together, these two channels generate a crash in the risky asset price that is much larger than the drop in fundamentals. The lower perceived crash risk after years with no crashes also means that the average excess return on the risky asset is low at precisely the moment when any crash that occurs would be especially large in size; moreover, it means that, in the event of a crash, some investors may default and banks may sustain large unexpected losses. Finally, the model shows how pre-crash warning signs can generate financial fragility. By reducing investors' optimism, warning signs also increase investors' uncertainty about their beliefs and thereby make them more likely to overreact to future bad news.

A central topic of study in economics is financial instability—crashes in financial markets, bank losses and failures, as well as the economic downturns that often accompany these events. One of the most striking findings to emerge from recent research on these issues is the close link between debt accumulation and subsequent instability. In a comprehensive analysis of financial crises over the past eight centuries across 66 countries, Reinhart and Rogoff (2009) show that debt accumulation during an economic boom often induces greater systemic risks than is initially apparent, and can be followed by a severe financial crash. Similarly, Mian and Sufi (2010, 2011, 2014) document a sharp rise in household leverage during the years before the Great Recession; Glick and Lansing (2010) and Jorda, Schularick, and Taylor (2011) demonstrate a close relation between the build-up of credit expansion and the severity of subsequent recessions; and Baron and Xiong (2014) find that bank credit expansion predicts a higher probability of a subsequent equity crash.

What is the origin of this debt-linked financial instability? A growing strand of empirical work highlights the importance of incorrect beliefs about crash risk: Coval, Jurek, and Stafford (2009) show that investors underestimated the probability and the correlations of mortgage defaults during the most recent credit boom; Foote, Gerardi, and Willen (2012) suggest that both investors and banks underestimate the likelihood of a potential crash event during economic booms; Coval, Pan, and Stafford (2014) find that suppliers of downside economic insurance products underestimate crash risk before a crash occurs; and Baron and Xiong (2014) discover that, although crash risk is significantly higher following bank credit expansions, the equity excess returns are nevertheless lower, even in the absence of crashes. Although these empirical findings all suggest that incorrect beliefs about crash risk may play a role in generating credit booms, asset bubbles, and price crashes, a formal dynamic theory that captures this narrative has yet to be developed.

In this paper, I develop a continuous-time equilibrium model that studies the impact of incorrect beliefs about crash risk on asset prices, portfolio decisions, and bank losses. There are two assets in the infinite-horizon economy: a risk-free asset with a fixed return; and a risky asset which is a claim to a stream of dividends that are subject to occasional crashes in fundamentals governed by a Poisson process with a constant likelihood (I use "likelihood" and "intensity" interchangeably hereafter). There are three types of agents: speculators, long-term investors, and banks. Both speculators and long-term investors trade in the asset markets. Banks, on the other hand, provide collateralized funding to speculators when these speculators want to take a levered position in the risky asset. Long-term investors exhibit a downward-sloping demand for the risky asset; their presence allows speculators to sell their risky asset holdings when they become too pessimistic.

Speculators—and their belief structure—are a primary focus of the paper. While the true crash intensity is constant, speculators have incorrect views about it—they believe that the crash intensity switches between a high-intensity state and a low-intensity state. This is the only difference between the speculators in my model and the agents in a model with full information; the speculators make proper Bayesian updates to their beliefs based on the observed crashes and are otherwise fully rational in their optimizations. Importantly, this one deviation drives many of the model implications.

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I first use the model to show that the magnitude of a price crash can sometimes be much larger than the magnitude of the crash in fundamentals. The amplification works through two channels. The first is a standard wealth channel: when speculators are highly levered, they respond to a crash by selling a portion of the risky asset to reduce leverage to the desired level, pushing the price further down. The second channel operates through beliefs: a crash occurrence provides evidence to speculators that the economy is more likely to be in the high-intensity state, causing them to increase their perceived likelihood of future crashes and to move their portfolios into the safe asset. There is a strong interaction between these two amplification channels: on the one hand, it is the underestimation of crash intensity that leads speculators to take on excessive leverage before a crash; on the other hand, the post-crash deterioration in beliefs makes the pre-crash debt accumulation a stronger force to push down the asset price. I show that, without incorrect beliefs about crash intensity, the wealth channel alone does not generate a strong amplification—with fully correct beliefs, speculators do not take on excessive leverage during the lead-up period, and they do not panic after observing a sequence of crash events.

I then show that the model generates a positive relation between the *size* of the credit expansion and the *severity* of a subsequent price crash. Following a long period without crashes, speculators' perceived likelihood of future crashes becomes lower than the true crash likelihood, driving them to take on excessive leverage—banks are willing to lend to speculators because they have the underlying asset as collateral. Through levered risky investment, the amount of speculative capital grows at a faster rate, adding fragility to the economy. Upon a crash, the wealth effect and the deterioration in beliefs together lead to severe deleveraging, and a larger amount of speculative capital associated with pessimistic beliefs further exacerbates the price crash. An amplified price crash has persistent effects on the economy because the net worth of speculators grows only slowly as their confidence recovers.

Consistent with the empirical results in Baron and Xiong (2014), the model also predicts that, even as the true crash risk rises following a credit expansion, the average excess return of the risky asset falls, and this is true even in the absence of a subsequent crash: with a lower perceived crash intensity following a credit expansion, strong demand from the speculators pushes up the equilibrium price of the risky asset and pushes down its average excess return.

In the model described so far, I have assumed that, while speculators have incorrect beliefs about crash *likelihood*, all agents in the economy hold correct beliefs about the *severity* of a potential crash in the risky asset price. Given this, no default occurs on the collateralized loans, and banks play the same role as a riskless technology. In the next part of the paper, I relax this assumption and allow for incorrect beliefs of speculators and of banks about crash severity. I endogenize these beliefs using a bounded rationality argument. In particular, I show that, if agents—banks and speculators—are unable to properly anticipate how other agents will act in the event of a crash, this can generate underestimation and overestimation of crash severity in different states of the world. With incorrect beliefs about crash severity, speculator defaults arise endogenously and banks can take unanticipated losses upon a crash. In the absence of a crash, the perceived

crash intensity decreases over time and excessive debt accumulation takes place. In such a levered economy, failure to fully assess the market-wide deleveraging causes banks to require an insufficient amount of collateral, which, in turn, leads to unanticipated losses for them when a crash occurs. In this way, incorrect beliefs about crash likelihood and incorrect beliefs about crash severity interact, and together generate a relation between banks' funding decisions in the lead-up period and the degree to which they are caught by surprise during financial crises. Indeed, a growing literature indicates that financial intermediaries may suffer from unanticipated losses during a financial crisis due to excessive lending in the preceding period (see, for example, Coval et al. (2009), Coval et al. (2014), and Baron and Xiong (2014)).

The model makes several additional predictions. First, the belief dynamics about crash risk naturally give rise to strong procyclical leverage, consistent with the empirical findings of procyclical leverage for households, financial intermediaries, and hedge funds documented in Mian and Sufi (2010), Adrian and Shin (2010), and Ang, Gorovyy, and van Inwegen (2012), respectively. Second, the model simultaneously matches the countercyclical true Sharpe ratios and procyclical perceived Sharpe ratios observed in the data (Lettau and Ludvigson (2010); Amromin and Sharpe (2013)).[1] Third, in equilibrium, return extrapolation arises endogenously in the model, matching the survey evidence analyzed by Vissing-Jorgensen (2004), Bacchetta, Mertens, and van Wincoop (2009), Amromin and Sharpe (2013), and Greenwood and Shleifer (2014). Finally, the model highlights the role of pre-crash warning signs in creating financial fragility. By reducing the level of optimism, warning signs increase speculators' uncertainty about their own beliefs and make them more likely to overreact to future crashes, causing larger price drops and bigger losses for banks and speculators.

In summary, my paper provides a unified theory for explaining both booms and busts; the nature and the magnitude of the boom drive the severity of the bust and the speed of the subsequent recovery. All of these results hinge on the incorrect beliefs about crash risk.

The paper is related to the general notion, proposed in Kindleberger (1978) and Minsky (1992), that prolonged economic booms lead to overoptimism and credit expansions, which introduce fragility into the financial system. My framework is also connected to the heterogeneous-belief models analyzed by Fostel and Geanakoplos (2008), Geanakoplos (2010), and Simsek (2013). However, my framework is fully dynamic while these models are, by and large, static. This is a crucial distinction: it is the dynamics of beliefs that play a key role in the implications of the current model. This paper also provides a micro-foundation for the neglected risk model of Gennaioli, Shleifer, and Vishny (2012): my model only needs the incorrect beliefs about crash risk to explain both credit expansions and severe price crashes, whereas the model of Gennaioli et al. (2012) needs both neglected risks and infinite risk aversion for investors to explain these results. In addition, this paper is related to the large literature in macroeconomics on business fluctuations and financial instability.[2] The principal difference between the current paper and this previous work is that I

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[1]Notice that, without the belief dynamics about crash risk, the standard wealth effect generates a countercyclical, rather than procyclical, pattern for leverage.

[2]For models of financial frictions, see Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Bernanke, Gertler, and Gilchrist (1999), He and Krishnamurthy (2012, 2013), and Brunnermeier and Sannikov (2014). For models of

study incorrect beliefs while these other papers study financial frictions, agency costs, or information production. My model setting is similar to the model of Xiong (2001). Nevertheless, on top of his framework, mine brings in aggregate crash events, collateralized funding, and the belief dynamics about crash risk that are central to my results.

Lastly, my model is related to the asset pricing literature on aggregate stock market behavior, and, in particular, to the time-varying rare disaster models of Gabaix (2012) and Wachter (2013). Three fundamental differences are worth noting. First, the disaster events in Gabaix (2012) and Wachter (2013) correspond to events that cause dramatic consumption drops in the economy—these are events that occur every 30 to 40 years; the crash events in my model are more frequent, corresponding to financial crashes and economic downturns. Second, in Gabaix (2012) and Wachter (2013), investors have fully correct beliefs about disaster risk, so the equilibrium risk premium is low when the true disaster intensity is low; however, in my model, investors have incorrect beliefs about crash risk, so the equilibrium risk premium is low when the *perceived* crash intensity is low—a moment at which actual crash risk is high. Last, the representative-agent approach in these models cannot address the empirical patterns on leverage, whereas leverage plays a crucial role in my model.

The paper proceeds as follows. In Section I, I lay out the basic elements of the model. In Section II, I first examine the implications of a benchmark model in which all agents have fully correct beliefs. Section III introduces and studies incorrect beliefs about crash *likelihood*. Section IV extends the model to allow for incorrect beliefs about crash *severity*. Section V concludes and suggests directions for future research. All proofs, numerical analyses, and discussion of some technical issues are in Appendices A and B.

I. General Framework

In this section, I lay out the basic structure of the model. The economy consists of two assets: a risky asset with a fixed per-capita supply of one; and a safe asset with a perfectly elastic supply and a constant interest rate r. The risky asset is a claim to a continuous dividend stream whose level evolves as

$$dD_t/D_t = g_D dt + \sigma_D d\omega_t - \kappa(dN_t - \lambda dt), \tag{1}$$

where g_D, σ_D, λ, and κ are all positive constants, ω_t is a one-dimensional Weiner process, and N_t is a Poisson process with intensity λ. On average, the dividend level grows at a rate of g_D with random small fluctuations and occasional reductions of κ percent when a Poisson event occurs. The Poisson event induces a crash in the asset price.

The equilibrium price of the risky asset is denoted P_t. Its evolution can be generically written as

$$dP_t/P_t = g_{P,t} dt + \sigma_{P,t} d\omega_t - \kappa_{P,t}(dN_t - \lambda dt), \tag{2}$$

risk-based funding constraints, see Danielsson, Shin, and Zigrand (2012) and Adrian and Boyarchenko (2012). For models of information production, see Gorton and Pennacchi (1990) and Gorton and Ordonez (2014).

where $g_{P,t}$, $\sigma_{P,t}$, and $\kappa_{P,t}$ are to be determined endogenously in equilibrium. I denote total return volatility at time t as $\bar{\sigma}_{P,t}$; from (2), $\bar{\sigma}_{P,t} = (\sigma_{P,t}^2 + \lambda \kappa_{P,t}^2)^{1/2}$.

There are three types of agents in the economy: speculators, long-term investors, and banks. Only speculators and long-term investors hold the risky asset—banks do not hold the risky asset themselves; they provide collateralized funding to speculators when the speculators want to take a levered position in the risky asset. I assume that, among the holders of the risky asset, speculators make up a fraction μ of the population and long-term investors, a fraction $1 - \mu$.

Long-term investors have a downward-sloping demand for the risky asset, namely

$$Q_t = (P_{F,t} - P_t)/k, \tag{3}$$

where $P_{F,t} \equiv D_t/(r - g_D)$ is the long-term investors' estimate of the fundamental value of the asset—the value determined by risk-neutral investors discounted at the interest rate without any risk compensation.[3] The presence of long-term investors provides a buyer for speculators to sell their risky investment to when they become very pessimistic.

Speculators—and their belief structure—are a primary focus of the paper. Speculators have incorrect views about the likelihood of crashes: while the true crash intensity is λ, they mistakenly believe that the latent crash intensity $\tilde{\lambda}_t$ follows a two-state Markov chain with high- and low-state intensity states, λ_h and λ_l, and update their expected crash intensity using Bayes' law based on past occurrences of crashes. These beliefs about crash intensity capture the notion that, after not observing a crash for a long time, crash events fade in speculators' memories, leading them to underestimate the likelihood of a potential crash event; and conversely that, after a sequence of crashes, crashes are salient to speculators, leading them to overestimate the likelihood of future crashes. These dynamics are consistent with the literature in psychology on the availability heuristic and memory biases; they are also consistent with the survey evidence on crash expectations and the empirical evidence on investors' risk taking behavior in financial markets.[4] Speculators may also have incorrect beliefs about the magnitude or severity of price declines in the event of a crash.

All speculators have time-additive log-utility preferences with a discount rate ρ.[5] At time 0, speculator i maximizes

$$\mathbb{E}_0^i \left[\int_0^\infty e^{-\rho t} \ell n(C_t^i) dt \right] \tag{4}$$

subject to her beliefs and wealth evolution as specified later.

The lending relationship between speculators and banks works as follows. Each speculator can either save by depositing her wealth in a bank or borrow from a bank by using the risky asset as collateral. For simplicity, both the deposit and lending rates are fixed exogenously at the interest rate r, and both deposit and collateralized funding last only for a period of dt—

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[3]$P_{F,t}$ is the standard price from the Gordon growth model when the discount rate is r. Using a different discount rate does not generate any qualitative change in my results.

[4]I discuss the belief structure and the evidence for it in detail in Section III.

[5]The log utility assumption is for simplicity; it allows me to solve for the equilibrium without solving the value function for speculators.

new contracts are formed thereafter. Denote banks' *perceived* severity of the potential asset price crash at time t as $\hat{\kappa}_{P,t}^B$. For each dollar invested at time t through collateralized funding, banks provide $(1 - \hat{\kappa}_{P,t}^B)(1 - rdt)$ dollars requiring $1 - \hat{\kappa}_{P,t}^B$ dollars back at $t + dt$, and speculators put down the remaining $1 - (1 - \hat{\kappa}_{P,t}^B)(1 - rdt)$ dollars. If a default occurs, speculators pay back $\min[1 - \hat{\kappa}_{P,t}^B, \theta(1 - \kappa_{P,t})]$, where $\theta \geq 1$ measures the ability of banks to seize speculators' personal wealth, and $\kappa_{P,t}$ stands for the true severity of the crash in the asset price. When $\theta = 1$, banks cannot seize speculators' personal wealth and take a loss of $(\kappa_{P,t} - \hat{\kappa}_{P,t}^B)/(1 - \hat{\kappa}_{P,t}^B)$ dollars on each dollar they lent to speculators if $\kappa_{P,t} > \hat{\kappa}_{P,t}^B$. With a higher θ, banks can seize some of speculators' personal wealth and therefore experience a smaller loss. In short, banks' beliefs about crash severity is the key determinant for collateralized funding; if banks underestimate the crash severity, they may lend too much to speculators and end up experiencing losses when a crash event occurs. Defaults are an issue in the model only in Section IV.

II.   The Rational Benchmark

In this section, I focus on a special case of the general framework in which both speculators and banks hold fully correct beliefs about both the crash intensity and the crash severity—in other words, all agents in the economy know both λ and $\kappa_{P,t}$. I refer to this special case as the rational benchmark.

The economy is characterized by two state variables: speculators' per-capita wealth, W_t, and the dividend level D_t. Given log preferences for speculators, the constant stochastic returns to scale for the dividend process, and the linear demand curve of the long-term investors, P_t must be homogeneous of degree one in D_t and W_t. As a result, without loss of generality, $P_t/D_t = l(x_t)$, where $x_t \equiv W_t/D_t$ measures speculators' wealth relative to fundamentals. In Appendix A.1, I show that x_t is the only state variable that governs the evolutions of $g_{P,t}$, $\sigma_{P,t}$, and $\kappa_{P,t}$.

The evolution of speculator i's wealth is

$$dW_t^i = -C_t^i dt + rW_t^i dt + w_t^i W_t^i [(g_{P,t} + D_t/P_t - r)dt + \sigma_{P,t} d\omega_t - \kappa_{P,t}(dN_t - \lambda dt)], \qquad (5)$$

with the restriction that $w_t^i \leq \kappa_{P,t}^{-1}$. Here, w_t^i is the fraction of wealth invested by speculator i in the risky asset, either through collateralized investing (when $w_t^i > 1$), through direct investing (when $0 \leq w_t^i \leq 1$), or through shorting (when $w_t^i < 0$). The upper bound for w_t^i is imposed by banks to ensure that their loans can be repaid after a crash.

Below I provide the definition of an equilibrium.

Definition 1. *An equilibrium in the rational benchmark is characterized by the price process $\{P_t\}$ and the consumption and portfolio decisions $\{C_t^i, w_t^i\}$ for each speculator i such that:*

1) Given the price process $\{P_t\}$ and the constant interest rate r, speculator i's consumption and portfolio decisions solve the maximization problem in (4) subject to her wealth evolution in (5);

2) On a per-capita basis, the market clearing condition for the risky asset

$$\mu W_t w_t + (1 - \mu)Q_t = P_t \tag{6}$$

is satisfied at each point in time.[6]

To understand the implications of the rational benchmark, I begin with the following lemma that characterizes the optimizing behavior of price-taking speculators.

Lemma 1. *In the rational benchmark, speculator i's optimal consumption stream is*

$$C_t^i = \rho W_t^i, \tag{7}$$

and her optimal risky asset investment w_t^i is the lower root of

$$g_{P,t} + \lambda \kappa_{P,t} + l^{-1} - r - w_t^i \sigma_{P,t}^2 - \frac{\lambda \kappa_{P,t}}{1 - w_t^i \kappa_{P,t}} = 0. \tag{8}$$

That is,

$$(w_t^i)^* = w_t = \frac{\kappa_{P,t} A_t + \sigma_{P,t}^2 - \sqrt{(\kappa_{P,t} A_t - \sigma_{P,t}^2)^2 + 4\lambda \kappa_{P,t}^2 \sigma_{P,t}^2}}{2\kappa_{P,t}\sigma_{P,t}^2}, \tag{9}$$

where $A_t \equiv g_{P,t} + \lambda \kappa_{P,t} + l^{-1} - r$ is the average excess return of the risky asset in the absence of a crash.

Proof. See Appendix A.1. ∎

Lemma 1 shows that, in the absence of crashes, the standard tradeoff between the average excess return $g_{P,t} + \lambda \kappa_{P,t} + l^{-1} - r$ and the return variance $\sigma_{P,t}^2$ determines the optimal investment in the risky asset. In addition to this mean-variance tradeoff, speculators are concerned about crash risk; this is captured by the last term in (8): $\lambda \kappa_{P,t}/(1 - w_t^i \kappa_{P,t})$. Compared to the no-crash case, crash risk lowers investors' risky asset investment. When the conditional expected excess return $g_{P,t} + l^{-1} - r$ is positive, the optimal investment w_t^i is between 0 and $\kappa_{P,t}^{-1}$, and when the conditional expected excess return is negative, speculators optimally choose to short the asset. Since speculators have correct beliefs about the crash severity $\kappa_{P,t}$, they will never choose to invest more than $\kappa_{P,t}^{-1}$ in the risky asset regardless of how high its expected excess return is: as w_t^i approaches $\kappa_{P,t}^{-1}$ from below, there is a non-zero probability for speculator wealth to drop to a level that is close to zero. Log-utility investors—and more generally any investor with an infinite marginal utility at zero wealth—are extremely averse to such situations and hence reduce leverage.

With individual investors' optimal portfolio and consumption decisions in hand, I now characterize the equilibrium asset-pricing implications of the rational benchmark.

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[6]In this paper, attention is restricted to the symmetric equilibrium.

Proposition 1. *In the rational benchmark, the Brownian volatility, the average growth rate of the risky asset in the absence of a crash, and the crash severity on the asset price are, respectively,*

$$\sigma_{P,t}(x_t) = \frac{1 - (l'/l)x_t}{1 - (l'/l)(c_0 l - c_1)}\sigma_D, \tag{10}$$

$$(g_{P,t} + \lambda\kappa_{P,t})(x_t) \equiv \bar{g}_{P,t}(x_t) = [1 - (l'/l)(c_0 l - c_1)]^{-1}\left\{g_D + \lambda\kappa + \sigma_D\sigma_{P,t} - \sigma_D^2\right.$$
$$+ (l'/l)x_t[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + (l^{-1} - r - \sigma_D\sigma_{P,t})(c_0 l - c_1)/x_t] \tag{11}$$
$$\left. + \tfrac{1}{2}(l''/l)[(l/l')(\sigma_{P,t} - \sigma_D)]^2\right\},$$

$$\kappa_{P,t}(x_t) = \frac{\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t}{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t}, \tag{12}$$

where

$$c_0 = (k + 1 - \mu)/(\mu k), \qquad c_1 = (1 - \mu)/(\mu k(r - g_D)), \tag{13}$$

and the fraction of speculator wealth invested in the risky asset is

$$w_t(x_t) = (c_0 l - c_1)/x_t. \tag{14}$$

The price-dividend ratio $l(x_t)$ is the solution to

$$\frac{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t]((c_0 l - c_1)/x_t - 1)}{(1 - \kappa)\{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}l(x_t)$$
$$= l\left(\frac{\lambda}{(1 - \kappa)\{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}x_t\right), \tag{15}$$

a second-order ordinary differential-difference equation after substitution of (10) through (12), with boundary conditions

$$\lim_{x_t \to \infty} l(x_t) = (r - g_D)^{-1}, \tag{16}$$

$$\lim_{x_t \to 0} l(x_t) = c_1/c_0, \qquad \lim_{x_t \to 0} l'(x_t) = w(0)/c_0, \tag{17}$$

where $w(0) \equiv \frac{\kappa A + \sigma_D^2 - \sqrt{(\kappa A - \sigma_D^2)^2 + 4\lambda\kappa^2\sigma_D^2}}{2\kappa\sigma_D^2}$ and $A = g_D + \lambda\kappa + (r - g_D)(1 + k/(1 - \mu)) - r$. The argument, sometimes omitted, of the function l and of its derivatives in Eqs. (10), (11), (12), (14), and (15) is x_t, except on the right-hand side of (15) where it is the post-crash value.

Proof. See Appendix A.1. ∎

Proposition 1 shows that the asset price and the portfolio and consumption decisions are fully characterized by the price-dividend ratio $l(x_t)$, its first and second derivatives, and the evolution of the fundamental dividend process D_t. From (10) and (14), the Brownian volatility can be written

as

$$\sigma_{P,t}(x_t) = \underbrace{\frac{1 - (l'/l)x_t}{1 - (l'/l)w_t x_t}}_{\text{wealth effect}} \sigma_D. \tag{18}$$

For a typical set of parameter values that I specify later, l increases monotonically in x_t: when speculator wealth is low, most of the risky asset is held by long-term investors who require a low price in order to hold the asset; when speculator wealth is high, their strong demand pushes up the asset price. Provided that l' is positive, the return volatility $\sigma_{P,t}$ is greater (less) than the fundamental volatility σ_D when speculators' risky asset portfolio weight w_t is greater (less) than one. This captures the standard wealth effect. When w_t is greater than one, speculators borrow money to invest more in the risky asset. In this case, a price drop caused by a negative dividend shock would drive up speculators' leverage ratio and induce them to delever. This deleveraging pushes the asset price further down and hence amplifies the return volatility. Conversely, when w_t is less than one, speculators only invest a fraction of their wealth in the risky asset. In this case, a price drop caused by a negative dividend shock leads speculators to rebalance by purchasing more of the risky asset, which in turn reduces the return volatility. As I will show in the numerical analysis, this wealth effect also drives the model implications for the crash severity on the asset price $\kappa_{P,t}$.

The boundary conditions on l are given in (16) and (17). As $x_t \to \infty$, speculator wealth is much larger than the dollar supply of the risky asset, and the asset market can clear only if the expected excess return converges to zero; this leads to the condition in (16). On the other hand, as $x_t \to 0$, the level of the asset price is mostly determined by long-term investors while the sensitivity of the asset price with respect to x_t is mostly affected by speculators' portfolio decisions, and this leads to the two conditions in (17).

The differential-difference equation in (15) reflects speculators' rational forecast of the crash severity. When making portfolio and consumption decisions, speculators correctly evaluate the crash severity by thinking through the impact of a crash on the asset price, on the wealth level of all agents, and on the future consumption of all agents. Eq. (15) cannot be solved analytically; I therefore resort to numerical methods. The argument of the function l in the second line of the equation in (15) includes a jump in the state variable with the jump size endogenously determined in equilibrium. Due to this complexity, the standard finite-difference approach is insufficient for solving the problem. I instead adopt a projection method with Chebyshev polynomials—a commonly used numerical procedure in applied economics. To confine the state variable x_t to the domain required for Chebyshev polynomials, $[-1, 1]$, I apply the monotonic transformation

$$z_t = (x_t - \gamma)/(x_t + \gamma), \tag{19}$$

where γ is a positive constant. I then define $h(z) \equiv l(x(z))$ and solve it numerically. I discuss the details of the numerical analysis in Appendix B.1.

Throughout the paper, I use the following default parameter values: $\mu = 0.5$, $r = 4\%$, $g_D = 1.5\%$, $\rho = 1\%$, $\sigma_D = 10\%$, $k = 0.5$, $\lambda = 0.2$, and $\kappa = 0.04$. Here $\lambda = 0.2$ means that, on average, crashes occur every 5 years, while $\kappa = 0.04$ means that the dividend level decreases by 4% every time a crash occurs.

In Figure 1, I plot the price-dividend ratio h, the crash severity κ_P, speculators' risky asset portfolio weight w, the conditional expected capital gain of the risky asset g_P, the Brownian volatility σ_P, as well as the total volatility $\bar{\sigma}_P$, all as functions of the transformed wealth-dividend ratio z.

[Place Figure 1 about here]

Figure 1 shows that the equilibrium price-dividend ratio decreases as speculators' per-capita wealth level decreases. On the one hand, speculators' dollar demand for the risky asset drops as their wealth level drops; on the other hand, the long-term investors will absorb a larger fraction of the asset supply only if the asset price decreases. The positive relation between the price-dividend ratio and speculator wealth explains the pattern for speculators' risky asset portfolio weight w. As speculator wealth decreases, so does their dollar demand for the risky asset; to induce the long-term investors to hold more of the risky asset, the price must fall, which in turn makes the risky asset more attractive to speculators and cause them to increase w.

To gain intuition on the behavior of the crash severity κ_P, the Brownian volatility σ_P, and the total volatility $\bar{\sigma}_P$, consider first the case when z moves away from one. In this case, speculator wealth drops from an extremely high level and the fraction of their wealth invested in the risky asset goes up from zero. With a small but positive w, the portfolio-rebalancing motive dampens the crash severity κ_P and the return volatilities σ_P and $\bar{\sigma}_P$. Consider next the case when z further decreases to the extent that speculators begin to take on debt, that is, w becomes greater than one. Here, a price reduction triggered by a dividend reduction causes their leverage to increase. Naturally, speculators would like to delever by selling some of their risky holdings to the long-term investors, and this pushes the price further down. This deleveraging serves as an amplification mechanism that pushes up κ_P, σ_P, and $\bar{\sigma}_P$. These patterns are consistent with the earlier discussion on the wealth effect. Consider last the case when z goes to -1. In this case, speculator wealth becomes very low and most of the risky asset is held by the long-term investors. As a result, deleveraging by speculators has little impact on the asset price and the price-dividend ratio tends to become constant, with κ_P, σ_P, and $\bar{\sigma}_P$ converging to κ, σ_D, and $(\sigma_D^2 + \lambda \kappa^2)^{1/2}$, respectively.

It is important to note that the rational bechmark is unable to match the empirical implications discussed in the Introduction. For example, the amplification in the rational benchmark is quite small—the maximum crash size in asset price is around 4.6%, only slightly higher than the 4% crash size in dividend. One key reason for this small amplification is that speculators' fully correct beliefs about the crash likelihood and severity allow them to prevent a large asset price drop and consumption reduction by (i) not taking on excessive leverage ex ante, and (ii) not panicking ex post after observing a sequence of crash events. As I will show in Sections III and IV,

10

allowing for incorrect beliefs about crash intensity and crash severity greatly increases the amplification. Another example is that as suggested in Figure 1, without the belief dynamics about crash risk, the standard wealth effect in the rational benchmark generates a countercyclical, rather than procyclical, pattern for leverage.

III. Incorrect Beliefs about Crash Likelihood

In this section, I study the case where speculators have incorrect beliefs about the likelihood of future crashes. Specifically, I assume that, as in the rational benchmark, the true crash intensity is a constant λ. However, speculators mistakenly believe that the latent intensity $\tilde{\lambda}_t$ follows a two-state Markov chain with a high-state intensity λ_h and a low-state intensity λ_l that satisfy $0 < \lambda_l < \lambda < \lambda_h$, and with the transition matrix[7]

$$
\begin{array}{c}
\phantom{\tilde{\lambda}_t = \lambda_h} \tilde{\lambda}_{t+dt} = \lambda_h \quad \tilde{\lambda}_{t+dt} = \lambda_l \\
\begin{array}{c} \tilde{\lambda}_t = \lambda_h \\ \tilde{\lambda}_t = \lambda_l \end{array}
\left(
\begin{array}{cc}
1 - q_h dt & q_h dt \\
q_l dt & 1 - q_l dt
\end{array}
\right),
\end{array}
\tag{20}
$$

where $q_h, q_l > 0$ are the intensities for the transition from the high state to the low state and from the low state to the high state, respectively.

This belief structure can be motivated by the availability heuristic, the notion that people assess the frequency of an event by the ease with which instances or occurrences can be brought to mind.[8] After a sequence of dramatic events, these events become salient to speculators, leading them to increase their estimate of the frequency of such events. As these events move into the distant past, however, it becomes difficult to retrieve them from memory, and as a result, speculators lower their estimate of the frequency of such events.

Empirical and survey evidence both lend support to the availability heuristic and the belief structure proposed above. Foote et al. (2012) present evidence to show that both investors and banks underestimate the likelihood of a potential crash event during economic booms. Coval et al. (2014) suggest that suppliers of downside economic insurance products neglect crash risk before a crash occurs as they charge an insufficient risk premium. Dessaint and Matray (2014) show that firms located close to a hurricane path become fearful after the event and increase the amount of corporate cash holdings even though the real liquidity risk remains unchanged, and also that this behavior reverses over time as the disaster recedes further back in time. The survey data collected by Robert Shiller—his data measures, for both individual and institutional investors, their level of confidence that there will be no stock market crash in the next six months—indicate that financial crises tend to spur depression fears, while subsequent recoveries and economic booms tend to drive

---

[7]The belief structure can be viewed as a continuous-time limit of the regime-switching belief structure proposed in Barberis, Shleifer, and Vishny (1998).

[8]See Tversky and Kahneman (1974), Kahneman and Frederick (2002), and Schwarz and Vaughn (2002) for detailed discussion of the availability heuristic.

the confidence level up.[9]

In my model, speculators use past occurrences of crashes up to time t as their information set to form beliefs about π_t, the probability that $\tilde{\lambda}_t$ equals λ_h. I will denote their beliefs by the perceived intensity $\lambda_t \equiv \pi_t \lambda_h + (1 - \pi_t)\lambda_l$. Solving the filtering problem for the belief evolution of λ_t gives[10]

$$
\begin{aligned}
d\lambda_t &= [q_l(\lambda_h - \lambda_t) - q_h(\lambda_t - \lambda_l) - (\lambda_h - \lambda_t)(\lambda_t - \lambda_l)]dt + \lambda_t^{-1}(\lambda_h - \lambda_t)(\lambda_t - \lambda_l)dN_t \\
&\equiv a(\lambda_t)dt + b(\lambda_t)(dN_t - \lambda_t dt).
\end{aligned}
\tag{21}
$$

Several observations about the λ_t process are worth noting. First, at any time, λ_t stays within (λ_m, λ_h) unless the prior beliefs fall outside this region. The lower bound λ_m is greater than λ_l because speculators know that, over any dt interval, there is a positive probability such that $\tilde{\lambda}_t$ can switch from λ_l to λ_h, and as a result their expected intensity λ_t will be higher than λ_l.[11,12] Second, both barriers, λ_m and λ_h, are inaccessible. For any $\lambda_t \in (\lambda_m, \lambda_h)$, λ_t deterministically trends downward unless a Poisson event occurs and pushes up its level: in the absence of Poisson events for a long period, λ_t decreases and gets asymptotically close to its lower barrier λ_m; conversely, after frequent Poisson events, λ_t gets pushed up towards λ_h. Lastly, upon a Poisson event, λ_t exhibits the largest jump in value if its pre-jump level is $(\lambda_h \lambda_l)^{1/2}$; with the expected intensity at this level, speculators are most unsure about the true value of $\tilde{\lambda}_t$ and therefore the amount of information carried by a crash event is the largest.

To prevent the model implications from being driven by speculators who are either too optimistic or too pessimistic on average, I impose the following parameter restriction:

$$
\lambda = \frac{q_l}{q_h + q_l}\lambda_h + \frac{q_h}{q_h + q_l}\lambda_l.
\tag{22}
$$

This ensures that, from speculators' perspective, the average λ_t in the long run is equal to the true intensity λ.

Given their estimate of the expected crash intensity, λ_t, speculators believe the dividend evolution is

$$
dD_t/D_t = (\hat{g}_{D,t} + \lambda_t \kappa)dt + \sigma d\hat{\omega}_t - \kappa dN_t,
\tag{23}
$$

where $\hat{g}_{D,t}$ is their perceived expected growth rate of the dividend and $d\hat{\omega}_t$ is their perceived Brownian shock. As the actual dividend can be observed, this evolution must match the true evolution

$$
dD_t/D_t = (g_{D,t} + \lambda \kappa)dt + \sigma d\omega_t - \kappa dN_t.
\tag{24}
$$

Because the focus of this paper is on the effects of incorrect beliefs about crash risk, I assume that

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[9]The Crash Confidence Index is at http://som.yale.edu/faculty-research/our-centers/international-center-finance/data.

[10]This is a special case of the filtering problem derived in Moreira and Savov (2014) without a noisy signal. The detailed derivation is in Appendix A.2.

[11]Here λ_m is given by $\{\lambda_h + \lambda_l + q_h + q_l - [(\lambda_h - \lambda_l + q_h - q_l)^2 + 4q_h q_l]^{1/2}\}/2$.

[12]As q_l goes to zero, λ_m converges to λ_l from above.

speculators' perceived Brownian shock equals the true shock $d\hat{\omega}_t = d\omega_t$. This means that

$$\hat{g}_{D,t} = g_D + \kappa(\lambda - \lambda_t). \tag{25}$$

That is, whenever the speculators are incorrect about crash intensity, they must make an offsetting error in their belief about the expected growth rate of the dividend.[13]

Applying a similar argument to the equilibrium price evolution gives[14,15]

$$\hat{g}_{P,t} = g_{P,t} + \kappa_{P,t}(\lambda - \lambda_t). \tag{26}$$

Eq. (26) is in line with the empirical findings of Vissing-Jorgensen (2004), Bacchetta et al. (2009), Amromin and Sharpe (2013), and Greenwood and Shleifer (2014): after a period of high realized returns due to the absence of crash events, speculators decrease their perceived likelihood of future crashes, and therefore increase their perceived expected return relative to the true expected return. In this specific sense, extrapolating the likelihood of crashes is one possible source of the return extrapolation we observe among real-world investors.

Using (26), the evolution of the equilibrium price can be written as

$$
\begin{aligned}
dP_t/P_t &= (\hat{g}_{P,t} + \lambda_t\kappa_{P,t})dt + \sigma_{P,t}d\omega_t - \kappa_{P,t}dN_t \\
&= (g_{P,t} + \lambda\kappa_{P,t})dt + \sigma_{P,t}d\omega_t - \kappa_{P,t}dN_t.
\end{aligned} \tag{27}
$$

Therefore, speculator i's wealth evolves as

$$
\begin{aligned}
dW_t^i &= -C_t^i dt + rW_t^i dt + w_t^i W_t^i[(\hat{g}_{P,t} + \lambda_t\kappa_{P,t} + D_t/P_t - r)dt + \sigma_{P,t}d\omega_t - \kappa_{P,t}dN_t] \\
&= -C_t^i dt + rW_t^i dt + w_t^i W_t^i[(g_{P,t} + \lambda\kappa_{P,t} + D_t/P_t - r)dt + \sigma_{P,t}d\omega_t - \kappa_{P,t}dN_t].
\end{aligned} \tag{28}
$$

The economy is now characterized by three state variables: speculators' perceived crash intensity λ_t, their per-capita wealth W_t, and the dividend level D_t. The assumptions that speculators have log-utility preferences, that the dividend process exhibits constant stochastic returns to scale, that the long-term investors' demand curve is linear in D_t and P_t, and that the belief structure in (20) is imposed on a unit-free quantity λ_t, imply that P_t is homogenous of degree one in D_t and W_t. Without loss of generality, then, $P_t/D_t = l(x_t, \lambda_t)$. In Appendix A.2, I show that the evolutions of $g_{P,t}$, $\sigma_{P,t}$, $\kappa_{P,t}$, w_t, and $\hat{g}_{P,t}$ are fully governed by the evolutions of λ_t and x_t.

For this model, the definition of an equilibrium is given below.

Definition 2. *An equilibrium is characterized by the price process $\{P_t\}$ and the consumption and*

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[13]Eq. (25) says that, after a period of high realized dividend growth due to lack of crashes, speculators become more optimistic about the expected dividend growth rate. In this sense, extrapolating the crash intensity can be transformed to extrapolating the dividend growth rate. Extrapolation of fundamentals is consistent with the empirical evidence documented by Lakonishok, Shleifer, and Vishny (1994), La Porta, Lakonishok, Shleifer, and Vishny (1997), and Greenwood and Hanson (2013).

[14]As in any continuous-time model, the quadratic variations of dP_t give the exact value of $\sigma_{P,t}$. In addition, in this section, all agents are assumed to have correct beliefs about $\kappa_{P,t}$.

[15]Eqs. (25) and (26) can both be viewed as variations of the Girsanov theorem.

portfolio decisions $\{C_t^i, w_t^i\}$ for each speculator i such that:

1) Given the price process $\{P_t\}$ and the constant interest rate r, speculator i's consumption and portfolio decisions solve

$$\max_{\{C_t^i, w_t^i \leq \kappa_{P,t}^{-1}\}_{t \geq 0}} \mathbb{E}_0^i \left[\int_0^\infty e^{-\rho t} \ell n(C_t^i) dt \right] \tag{29}$$

subject to her belief evolution in (21) and her wealth evolution in (28);

2) On a per-capita basis, the market clearing condition for the risky asset

$$\mu W_t w_t + (1 - \mu) Q_t = P_t \tag{30}$$

is satisfied at each point in time.

To gain some intuition for the model's implications, I first summarize speculators' optimizing behavior in the following lemma.

Lemma 2. *For the model in which speculators hold incorrect beliefs about the likelihood of future crashes, speculator i's optimal consumption stream is*

$$C_t^i = \rho W_t^i \tag{31}$$

and her optimal risky asset investment w_t^i is the lower root of

$$\hat{g}_{P,t} + \lambda_t \kappa_{P,t} + l^{-1} - r - w_t^i \sigma_{P,t}^2 - \frac{\lambda_t \kappa_{P,t}}{1 - w_t^i \kappa_{P,t}} = 0. \tag{32}$$

That is,

$$(w_t^i)^* = w_t = \frac{\kappa_{P,t} A_t + \sigma_{P,t}^2 - \sqrt{(\kappa_{P,t} A_t - \sigma_{P,t}^2)^2 + 4\lambda_t \kappa_{P,t}^2 \sigma_{P,t}^2}}{2\kappa_{P,t} \sigma_{P,t}^2}, \tag{33}$$

where $A_t \equiv \hat{g}_{P,t} + \lambda_t \kappa_{P,t} + l^{-1} - r = g_{P,t} + \lambda \kappa_{P,t} + l^{-1} - r$ is the average excess return of the risky asset in the absence of a crash.

Proof. See Appendix A.2. ∎

As in Lemma 1, the mean-variance-crash risk tradeoff still drives the portfolio decisions. The only difference is that speculators' beliefs about crash likelihood are now incorrect. When the perceived crash intensity λ_t is lower (higher) than the true crash intensity λ, speculators invest too much (too little) in the risky asset.

In the proposition below, I present the analytical results that describe the asset pricing implications of the model.

Proposition 2. *For the model in which speculators have incorrect beliefs about crash likelihood, the Brownian volatility, the average growth rate of the risky asset in the absence of a crash, and*

the severity in the risky asset price are, respectively,

$$\sigma_{P,t}(x_t, \lambda_t) = \frac{1 - (l_x/l)x_t}{1 - (l_x/l)(c_0 l - c_1)}\sigma_D, \tag{34}$$

$$(g_{P,t} + \lambda\kappa_{P,t})(x_t, \lambda_t) \equiv \bar{g}_{P,t}(x_t, \lambda_t) = [1 - (l_x/l)(c_0 l - c_1)]^{-1}\left\{g_D + \lambda\kappa + \sigma_D\sigma_{P,t} - \sigma_D^2\right.$$

$$+ (l_x/l)x_t[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + (l^{-1} - r - \sigma_D\sigma_{P,t})(c_0 l - c_1)/x_t] \tag{35}$$

$$\left. + (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)[(l/l_x)(\sigma_{P,t} - \sigma_D)]^2\right\},$$

$$\kappa_{P,t}(x_t, \lambda_t) = \frac{\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t}{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t}. \tag{36}$$

The fraction of speculator wealth invested in the risky asset is

$$w_t(x_t, \lambda_t) = (c_0 l - c_1)/x_t. \tag{37}$$

The price-dividend ratio $l(x_t, \lambda_t)$ is the solution to

$$\frac{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t]((c_0 l - c_1)/x_t - 1)}{(1 - \kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}l(x_t, \lambda_t)$$

$$= l\left(\frac{\lambda_t}{(1 - \kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}x_t, \lambda_t + b(\lambda_t)\right), \tag{38}$$

a second-order partial differential-difference equation after substitution of (34) through (36), with boundary conditions

$$\lim_{x_t \to \infty} l(x_t, \lambda_t) = [r - \lim_{x_t \to \infty} \hat{g}_{P,t}(x_t, \lambda_t)]^{-1} \equiv m(\lambda_t), \tag{39}$$

$$\lim_{x_t \to 0} l(x_t, \lambda_t) = c_1/c_0, \qquad \lim_{x_t \to 0} l_x(x_t, \lambda_t) = w(0, \lambda_t)/c_0, \tag{40}$$

for $\forall \lambda_t \in (\lambda_m, \lambda_h)$, where $w(0, \lambda_t) \equiv \frac{\kappa A + \sigma_D^2 - \sqrt{(\kappa A - \sigma_D^2)^2 + 4\lambda_t\kappa^2\sigma_D^2}}{2\kappa\sigma_D^2}$, and the function $m(\lambda_t)$ is determined by

$$m'(\lambda_t)[a(\lambda_t) - \lambda_t b(\lambda_t)] = m(\lambda_t)\left(\lambda_t - g_D - \lambda\kappa - m^{-1}(\lambda_t) + r - \frac{(1 - \kappa)\lambda_t m(\lambda_t + b(\lambda_t))}{m(\lambda_t)}\right). \tag{41}$$

The arguments, sometimes omitted, of the function l and of its derivatives in Eqs. (34) to (38) are x_t and λ_t, except on the right-hand side of (38) where they are the post-crash values.

Proof. See Appendix A.2. ∎

Proposition 2 shows that the asset pricing implications of the model are governed by the price-dividend ratio $l(x_t, \lambda_t)$, its first and second derivatives, the evolution of the fundamental dividend process D_t, and the belief dynamics of λ_t.[16] More important, the interaction between the wealth

---

[16]As both λ_h and λ_l converge to λ, Proposition 2 reduces to Proposition 1, which characterizes the equilibrium for

effect and speculators' belief dynamics drives many of the model's implications.

From (34) and (37), the Brownian volatility is

$$
\sigma_{P,t}(x_t, \lambda_t) = \underbrace{\frac{1 - (l_x/l)x_t}{1 - (l_x/l)w_t x_t}}_{\text{wealth effect and belief effect}} \sigma_D. \tag{42}
$$

From Lemma 2, the multiplier in (42) is similar to that in (18) but now incorporates the effects of both wealth and beliefs. When the perceived crash likelihood is lower than the true crash likelihood, speculators to take on excessive leverage. Then, from (42), upon a negative Brownian shock to the dividend, higher leverage will lead to a larger percentage loss of speculator wealth and hence a stronger deleveraging effect. This in turn results in a larger amplification of the Brownian volatility. Therefore, the multiplier in (42) is jointly determined by the standard wealth effect and the belief effect.

For the crash severity on the equilibrium asset price, Eqs. (36), (37), and (38) imply

$$
\frac{1 - \kappa_{P,t}(x_t, \lambda_t)}{1 - \kappa} l(x_t, \lambda_t) = l \left(\underbrace{\frac{1 - w_t \kappa_{P,t}(x_t, \lambda_t)}{1 - \kappa} x_t}_{\substack{\text{wealth effect} \\ \text{and belief effect}}}, \underbrace{\lambda_t + b(\lambda_t)}_{\text{belief effect}} \right). \tag{43}
$$

The post-crash price-dividend ratio can be determined in two ways. First, given that the price level is reduced by $\kappa_{P,t}$ and the dividend level is reduced by κ upon a crash, the post-crash price-dividend ratio must equal its pre-crash level multiplied by $(1 - \kappa_{P,t})/(1 - \kappa)$, as suggested by the expression on the left-hand side of (43). Second, given that the price-dividend ratio l is a function of x_t and λ_t, changes in l must come from changes in x_t and λ_t, as suggested by the expression on the right-hand side of (43). By linking these two expressions, Eq. (43) shows that the equilibrium crash severity is jointly determined by the portfolio and funding decisions made by speculators and banks before a crash *and* by their reactions to the crash. The longer the lead-up period is, the more confident speculators become in terms of underestimating the likelihood of future crashes. Along with their optimistic beliefs, less wealthy speculators borrow heavily through collateralized funding so that they can invest more in the risky asset. As banks have the risky asset as collateral, they are willing to lend to speculators. This results in a credit boom and a more vulnerable economy. Upon a crash event, the higher leverage taken by the speculators in the pre-crash period causes more deleveraging that pushes down the price-dividend ratio and pushes up the crash severity—a higher w_t leads to a sharper decline in the state variable x_t on the right-hand side of (43), which in turn contributes to a larger crash severity $\kappa_{P,t}$—the change in speculator wealth is jointly determined by the wealth effect and the belief effect. In addition to causing the deleveraging process just described, the crash event also affects speculators' beliefs: their perceived crash likelihood increases from λ_t to $\lambda_t + b(\lambda_t)$. More pessimistic beliefs cause speculators to reduce risk taking and sell the risky asset,

the rational benchmark.

and this further exacerbates the decline of the asset price—the jump in λ_t on the right-hand side of (43) leads to a larger $\kappa_{P,t}$.

It is important to note that the second channel of deterioration in beliefs feeds back to the first deleveraging channel—a higher $\kappa_{P,t}$ caused by a sudden increase in λ_t further lowers speculators' post-crash wealth, and this, as a result, leads to additional deleveraging and pushes the asset price further down. It is also worth noting that the second belief channel and its feedback to the first deleveraging channel are key determinants only of the crash severity, but not of the Brownian volatility—in the absence of a crash, small fluctuations in the dividend do not cause speculators to increase their perceived crash likelihood. This is the reason why incorrect beliefs about crash likelihood greatly amplify the crash severity but not the Brownian volatility.

As for the rational benchmark, I use a projection method with Chebyshev polynomials to solve the differential-difference equation in (38), with the same change of variables $z_t \equiv (x_t - \gamma)/(x_t + \gamma)$ and with $h(z_t, \lambda_t) \equiv l(x(z_t), \lambda_t)$. The details of the numerical procedure are laid out in Appendix B.2. To examine the model's *quantitative* implications, I leave the parameter values provided in Section II unchanged and further specify the parameter values for the belief structure in (20): I set $\lambda_l = 0.025$, $\lambda_h = 1$, and $q_h = 0.025$, and derive $q_l = 0.005$ from the parameter restriction in (22). I provide some motivation for these parameter values when discussing the time-series properties of the model.

[**Place Figure 2a and Figure 2b about here**]

Figures 2a and 2b present 3-D and 2-D graphs, respectively, for the price-dividend ratio h, the crash severity $\kappa_{P,t}$, speculators' portfolio weight in the risky asset w_t, the Brownian volatility $\sigma_{P,t}$, the true expected capital gain of the risky asset $g_{P,t}$, and the perceived expected capital gain $\hat{g}_{P,t}$, all as functions of speculators' perceived crash intensity λ_t and the transformed wealth-dividend ratio z_t.

Figures 2a and 2b show that the price-dividend ratio h generally increases as λ_t decreases; speculators push up the asset price as their perceived likelihood of future crashes decreases. The price-dividend ratio also increases as z_t increases; as speculators become wealthier, their demand for the risky asset rises, and this pushes the asset price up.[17] By comparing Figures 2a and 2b with Figure 1, one can quantify the overvaluation and the undervaluation caused by the incorrect beliefs about crash likelihood. For instance, when $x_t = \infty$, speculator wealth is very high. In this case, the price-dividend ratio is 48.8 if $\lambda_t = \lambda_m$, 22% higher than its value in the rational benchmark; and if $\lambda_t = \lambda_h$, the price-dividend ratio is 27.1, 32% lower than its value in the rational benchmark.

---

[17]The validity of this statement depends on parameter values. In my example, the downward pressure on the asset price caused by speculators' pessimism is generally smaller compared to the downward pressured caused by the long-term investors when they are the only investors in the economy. This is because of two reasons. First, the value set for parameter k in Eq. (3) is such that the long-term investors have inelastic demand with respect to price changes, so they require a very low price to hold the entire supply of the risky asset. Second, even when speculators' perceived likelihood of future crashes is high, they know that their beliefs tend to revert back, and this puts an upper bound on the extent to which they want to push down the asset price. For a higher λ_h, a lower q_h, or a smaller k, the price-dividend ratio may decrease in z_t.

When $x_t = 10$, speculator wealth is very low. In this case, the price-dividend ratio is 25 if $\lambda_t = \lambda_m$, 3% higher than its value in the rational benchmark; and if $\lambda_t = \lambda_h$, the price-dividend ratio is 20.1, 18% lower than its value in the rational benchmark. The asymmetry in size between the overvaluation and the undervaluation is primarily due to the fact that crashes are infrequent so the true crash intensity λ is small; with λ closer to λ_m than λ_h, the highest degree of pessimism is larger than the highest degree of optimism.

The crash severity $\kappa_{P,t}$ is hump-shaped with respect to λ_t for any given level of speculator wealth. On the one hand, when speculators' perceived likelihood of future crashes is very low, seeing a crash will not cause them to become much more pessimistic because their uncertainty about their own beliefs is low; that is, the revision in beliefs, $b(\lambda_t)$, is small when λ_t is close to λ_m. Therefore, the belief effect characterized in (43) is not very strong. On the other hand, when speculators' perceived likelihood of future crashes is very high, seeing another crash is consistent with their current beliefs, so speculators will also not become much more pessimistic—again, the belief effect in (43) is not very strong when λ_t is close to λ_h. At an intermediate level of λ_t, however, speculators are quite uncertain about their own beliefs and therefore the information carried by a crash event is large.[18]

The crash severity $\kappa_{P,t}$ increases in speculator wealth for medium and high values of λ_t, but is hump-shaped in speculator wealth for low values of λ_t: a higher wealth level for speculators means that changes in their beliefs have a larger impact on the asset price, while a lower wealth level leads to higher leverage and hence stronger deleveraging if a crash occurs; it is only when λ_t is low that speculators take on sufficiently high leverage so that deleveraging plays a significant role in determining $\kappa_{P,t}$.

It is important to note that, in the model, the belief and the wealth effects combined generate a much stronger amplification than does the standard wealth effect alone. When $\lambda_t = 0.20$ and when speculator wealth is high—for example, when $x_t = \infty$—the crash severity can be as high as 37.4%; this is more than nine times as high as the dividend reduction of 4%. Even when speculator wealth is low—for example, when $x_t = 10$—the crash severity is 18.9% when $\lambda_t = 0.20$; this is still more than 4.5 times as high as the dividend reduction. In this case, the debt-to-equity ratio is 61.3%; collateralized funding effectively increases the amount of capital controlled by the speculators and the impact of their beliefs on the asset price. As shown in (43), high leverage in the pre-crash period and the post-crash deterioration in beliefs together greatly amplify the equilibrium crash severity.

The risky asset portfolio weight w_t increases as λ_t decreases for any given level of speculator wealth; all else equal, a lower perceived likelihood of future crashes causes speculators to invest more in the risky asset. Quantitatively, a large variation in w_t takes place as λ_t changes—for example, $w(0, \lambda_t)$ in (40) shows that, when speculators have little wealth, their portfolio weight w_t rises from −61% to 316% as λ_t drops from λ_h to λ_m.

For low and medium values of λ_t, the portfolio weight w_t increases as speculator wealth de-

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[18]This is related to the uncertainty of beliefs analyzed in Veronesi (1999)

creases, but is hump-shaped in speculator wealth for high values of λ_t. At lower levels of speculator wealth, the downward pressure on the asset price created by the long-term investors increases, and therefore the dividend-yield becomes higher. This generally makes the risky asset more attractive to speculators and hence increases the portfolio weight they put on the asset. For a high value of λ_t, however, a counterforce becomes important. With lower wealth, speculators have less capacity to push up the asset price in later periods as their pessimism level decreases over time. Given this, speculators' perceived expected capital gain of the risky asset decreases as their wealth level decreases; this reduces the portfolio weight speculators put on the risky asset.

Generally, the Brownian volatility $\sigma_{P,t}$ has an inverse S-shaped relation with speculator wealth, and this relation gets stronger as speculators' perceived crash likelihood decreases. With a lower value of λ_t, speculators invest more in the risky asset through collateralized funding, and the deleveraging effect captured in (42) becomes larger upon a negative Brownian shock on the dividend. By comparing Figures 2a and 2b with Figure 1, one can quantify the impact the belief dynamics in (21) have on $\sigma_{P,t}$. For example, when $x_t = 10$, speculators with fully correct beliefs take a debt-to-equity ratio of 76% and $\sigma_{P,t} = 11.46\%$. Speculators with incorrect beliefs about crash likelihood take a debt-to-equity ratio of 101% if $\lambda_t = \lambda_m$, and $\sigma_{P,t} = 12.23\%$. Note that the amplification the belief effect creates on the Brownian volatility is much smaller than the amplification it creates on the crash severity: in the model, small Brownian shocks do not affect speculators' beliefs about the likelihood of future crashes, but Poisson shocks can cause large deterioration in beliefs that are detrimental to the asset price.[19]

The true expected capital gain of the risky asset $g_{P,t}$ is primarily driven by the rate at which speculators' belief about λ_t trends down in the absence of crashes—this is measured by $a(\lambda_t) - \lambda_t b(\lambda_t)$—and by the sensitivity of the price-dividend ratio to belief changes—this is measured by l_λ / l. At an intermediate level of λ_t, speculators' perceived crash likelihood decreases at a fast rate and the impact of the belief dynamics on the asset price is large. In this case, later periods' price will be substantially pushed up by speculators' decreasing level of perceived crash likelihood, and this in turn pushes up the true expected capital gain of the risky asset in the current period.

It is important to note that the *perceived* expected capital gain of the risky asset $\hat{g}_{P,t}$ and the true expected capital gain $g_{P,t}$ exhibit quite different patterns. The perceived quantities are the ones that determine speculators' portfolio and consumption decisions. For either a low or a high level of wealth, speculators' perceived expected capital gain increases as λ_t decreases. When speculator wealth is low, the asset price is, by and large, determined by the long-term investors, and the expected capital gain of the asset price $g_{P,t}$ stays approximately constant at the fundamental growth rate g_D. The perceived expected capital gain $\hat{g}_{P,t}$ is therefore higher than g_D when speculators underestimate the crash likelihood, and is lower than g_D when speculators overestimate the crash likelihood—this results in a negative relation between $\hat{g}_{P,t}$ and λ_t when x_t is low. When speculator wealth is high, their holding of the risky asset is low, so in equilibrium, the perceived expected

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[19]A different reason that limits the impact of beliefs on σ_P is the log preferences of speculators—if speculators' relative risk aversion is much lower than one, then σ_P will become more responsive to λ_t because high confidence that crashes are unlikely to occur would stimulate risk taking to a larger extent.

excess return must be close to zero. As a result, if λ_t is low, the dividend yield is low, and the perceived expected capital gain must be high; conversely, if λ_t is high, the dividend yield is high, and the perceived expected capital gain must be low—this again results in a negative relation between $\hat{g}_{P,t}$ and λ_t when x_t is high.[20] With either a high or a low wealth level, speculators have little concern about the impact of their beliefs on the crash severity: when their wealth level is high, speculators invest mostly in the safe asset; and when their wealth level is low, speculators' beliefs have little impact on the asset price. However, at an intermediate wealth level, speculators have a high concern about the impact of their beliefs on the crash severity because, in this case, a crash on the asset price may significantly reduce their future consumption. Therefore, in order to induce more risk taking, speculators' perceived expected capital gain must be high when the crash severity is high, that is, when λ_t is at an intermediate level—this results in a hump-shaped relation between $\hat{g}_{P,t}$ and λ_t when x_t is at an intermediate level. The same reasoning also generates a hump-shaped relation between $\hat{g}_{P,t}$ and x_t when λ_t is at an intermediate level—a level at which a new crash has the greatest effect on speculators' beliefs.

Following a sequence of crashes, speculators have pessimistic beliefs and tend to disinvest in the risky asset. In the absence of new crashes, pessimism fades over time; this raises speculators' perceived expectations of the capital gain on the risky asset. When financially constrained speculators believe that crashes are sufficiently unlikely to occur, they begin to take levered position through collateralized funding; this helps them to accumulate wealth at a faster rate. At an intermediate level of λ_t, the higher amount of speculative capital pushes up the perceived capital gain as described in the previous paragraph. As time passes and speculators become overly optimistic, "leaning against the wind" by the long-term investors prevents the true capital gain from becoming too low, and this again helps to keep the perceived capital gain high. This line of argument suggests that the model can capture the survey evidence on return expectations documented in Greenwood and Shleifer (2014): in surveys, investors' expectations about the market return increase after stock prices have previously risen and decrease after stock prices have previously fallen. The model suggests one source of extrapolative beliefs: investors tend to underestimate the likelihood of extreme events after not seeing them for a long time, and tend to overestimate the likelihood of such events once they observe a few of them in a row.

Figures 2a and 2b provide a state-space view of the model by plotting the values of various quantities of interest in any state of the world characterized by x_t and λ_t. To fully understand the model implications, I also examine how the asset price, the portfolio and consumption decisions, and investors' beliefs and wealth evolve over time. Below I analyze some important time-series properties of the model.

[**Place Figure 3a and Figure 3b about here**]

In Figures 3a and 3b, I present the time-series implications of the model given the initial values of the two state variables and a simulated dividend process in the subsequent periods. This analysis

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[20]At $x_t = 0$, the exact relation $\hat{g}_{P,t} = g_D + \kappa(\lambda - \lambda_t)$ holds; at $x_t = \infty$, the exact relation $\hat{g}_{P,t} = r - m^{-1}(\lambda_t)$ holds.

proceeds in four steps. First, I set the time span and the initial states for the time series by using two pieces of data. The Crash Confidence Index provided by Robert Shiller suggests that, for both individual and institutional investors, their perceived crash intensity peaked in November 2002 after the market bottomed out in October 2002. Therefore, I choose a ten-year time span from the beginning of November 2002 to the end of 2012 with a high initial value for λ_t of 0.5. Empirical studies also suggest that, for both U.S. households and hedge funds, leverage increased significantly in the lead-up period to the Great Recession. Therefore, I set a low initial value for x_t of 5 so that financially constrained speculators in the model would like to take levered positions in the risky asset once their expectations on the crash intensity become sufficiently low. Second, I preselect a few dates—dates that mark the periods during which dramatic events or news hit the market—for the Poisson shocks in the model. These dates include July 16th 2007, October 15th 2008, and August 8th 2011.[21] Third, I discretize Eq. (1) and simulate a single dividend process with a step size of a day, with an initial level of 10, and with three Poisson shocks on the three dates just described. Lastly, I use the model solution from Proposition 2 to generate model-implied time series of the equilibrium asset price, the crash severity, the belief dynamics of expected crash severity, the risky asset portfolio weight, the perceived expected excess return, and the true expected excess return, among other quantities.

Below I detail the time-series properties of the model by discussing a number of its empirical predictions that are consistent with the data.

A. Procyclical Leverage and Amplification of Crash Severity

Many empirical studies document that, for both individual and institutional investors, leverage exhibits a procyclical pattern. For example, Mian and Sufi (2010, 2011) show a striking rise in household leverage during the years before the Great Recession; Adrian and Shin (2010) present evidence of financial intermediaries actively adjusting their balance sheets in such a way that leverage is high during booms and low during busts; and Ang et al. (2012) document that hedge funds reduce leverage during financial downturns. Importantly, rising debt is a strong predictor of the likelihood and the severity of subsequent crashes: Baron and Xiong (2014) show that bank credit expansion significantly predicts a higher probability of subsequent equity crashes, while Glick and Lansing (2010) and Jorda et al. (2011) demonstrate a close relation between credit expansions and the severity of subsequent recessions.

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[21] July 16th 2007 marks the failure of two Bear Stearns-run hedge funds. This event carries the notion of warning signs embedded in a sequence of news throughout 2007 concerning the economic outlook; notably, the aggregate monthly earnings in the U.S. started to decrease after July 2007. Around October 15th 2008 was a hectic period with the financial sector being dysfunctional and with mounting concerns of a recession: a market panic on September 29th was triggered by the failure of the House of Representatives to pass a $700 billion bank bailout bill; a steep market decline on October 9th was caused by negative news about the auto industry; a significant Dow loss on October 15th was precipitated by Bush's announcement of the largest government intervention in the banking sector since the Great Depression; a big drop on October 22nd was fueled by increasing fears of a global recession; and a sharp decline on December 1st was sparked by the NBER's official announcement of a recession in the U.S. The stock market losses topped one trillion dollars on August 8th 2011 because of the downgrade of the U.S. federal government, four days after a large Dow drop triggered by fears of a double-dip recession and by worries about the European debt crisis.

The time-series patterns in Figures 3a and 3b suggest that my dynamic theory generates a strong procyclical leverage cycle for speculators. In the model, from November 2002 to the end of 2006, speculators' debt-to-equity ratio rises from 6.9% to 98.7% and their per-capita debt level accordingly increases from 3.4 to 132.5. A sequence of warning signs clustered in mid-July of 2007 leads to substantial deleveraging; speculators cut their borrowing from 95.1% to 26.5% of their wealth so that the debt level drops from 150.3 to 29.7. During the financial turmoil after the bankruptcy of Lehman Brothers in September 2008, speculators fully exit the credit market, with only 22.7% of their wealth invested in the risky asset. As the market stabilizes and recovers over time, speculators gradually increase their risk-taking until fears of a double-dip recession strike the market again in August of 2008, with speculators' risky asset portfolio weight plummeting from 150.8% to 8.5% and their debt level dropping from 87.1 to zero.

Consistent with the data, Figures 3a and 3b also suggest that the severity of the asset price crashes becomes larger after the credit expansion. In time order, the three crash events in the time series reduce the equilibrium price by 14.2%, 18.3%, and 16.3%, respectively; these price reductions are much larger than the dividend reductions of 4%. It is worth noting that this model quantifies the impact of higher leverage on the equilibrium crash severity—the crash severity plot in Figure 3a indicates that, if a Poisson event had hit the market at the beginning of November 2002 before the credit boom, the crash size would have been 10.7%, 42% lower than the largest price reduction in mid-October of 2008.

The fundamental reason why the model generates procyclical leverage cycle is the incorrect beliefs about crash likelihood. Indeed, as pointed out by Danielsson et al. (2012), the standard wealth effect alone, as analyzed in Xiong (2001), would generate *countercyclical* leverage: during booms, when investors are wealthy, higher dollar demand for the risky asset tends to push up the asset price and push down the Sharpe ratio, making it unattractive for investors to take on high leverage; conversely, during busts when investors are poor, lower dollar demand tends to push down the asset price and push up the Sharpe ratio, inducing investors to take on high leverage. In the model, however, a countervailing force arises from the belief dynamics; instead of a mean-variance tradeoff, speculators face a mean-variance-perceived crash risk tradeoff when making their portfolio decisions. After a sequence of crash events, the high expected crash intensity perceived by speculators cause them to reduce risk taking. As the crash events go into the distant past, the perceived crash risk decreases and leverage begins to rise. Toward the end of a bubble period, the debt-to-equity ratio tends to flatten out: on the one hand, with high leverage and good market conditions, speculator wealth grows at a faster rate than the asset price, making the wealth effect a stronger force, one that reduces leverage; on the other hand, there is an upper limit to speculators' degree of optimism. Upon new crashes, speculators become pessimistic and their wealth level drops significantly. The belief channel becomes the dominant force again, and a new cycle begins.

The belief dynamics are also the reason why the model yields a close relation between leverage and the crash severity. As summarized in Eq. (43), the standard wealth effect and the belief dynamics jointly determine the size of a crash; lower estimation of the crash intensity induces

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higher leverage, and this in turn means that a crash event and the associated belief deterioration have a larger impact on the asset price through more severe deleveraging.

B. Credit Expansion and Subsequent Excess Returns

Baron and Xiong (2014) show that credit expansion not only predicts future crash risk, but is also followed by *lower* equity excess returns, with or even without the presence of crash events in the subsequent periods. This finding presents a challenge to existing theories with fully correct beliefs because higher crash risks should be compensated by higher equity premia. However, the incorrect beliefs about crash intensity in this model naturally generate these empirical patterns. Indeed, Figure 3b suggests that both the true expected excess return and the average excess return in the absence of crash events are lower following a significant credit expansion: a credit expansion in the model is driven by an increasing degree of crash risk neglect, and it is exactly the crash risk neglect that drives down the true excess return.

C. Countercyclical Sharpe Ratios

Empirical evidence suggests that the market Sharpe ratio is countercyclical (see Lettau and Ludvigson (2010) for a detailed discussion of this topic). Figure 3b shows that the model generates such a countercyclical pattern, with the driving force being the sharp rise of the expected excess return during crises: on the one hand, the price reduction is much larger than the dividend reduction upon a crash and therefore the dividend yield spikes up; on the other hand, as the economy begins to recover, investor pessimism decays and their dollar demand for the risky asset rises, lifting up the asset prices in later periods, and, as a result, pushing up the current period's expected capital gain as well.

It is important to notice that, in my framework, even though the true Sharpe ratios are countercyclical, the perceived Sharpe ratios are strongly procyclical, consistent with the procyclical leverage cycle discussed above—the time-series correlation between the perceived Sharpe ratios and the true Sharpe ratios in Figure 3b is -0.78. Using survey data, Amromin and Sharpe (2013) show that the Sharpe ratios perceived by households are indeed procyclical. The contrast between these two time series stems from the incorrect beliefs about crash intensity; as I am about to discuss, although the true expected excess return spikes up during crashes, the perceived expected excess return drops significantly. [22]

D. Extrapolative Expectations

Survey evidence analyzed by Vissing-Jorgensen (2004), Bacchetta et al. (2009), Amromin and Sharpe (2013), and Greenwood and Shleifer (2014) suggests that many investors—both individual and institutional investors—hold extrapolative expectations, believing that the stock market will keep rising after rising in the past, and falling after falling in the past.

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[22]Any asset pricing model with fully correct beliefs will be unable to match both patterns.

The time series of perceived expected excess return in Figure 3b suggests that return extrapolation arises naturally in the model. After a sequence of crashes, the asset price goes down and investors become overly pessimistic, revising down their expected excess returns amid fears of future crashes. As historic crash events get into the distant past, the asset price rises and investors become more and more optimistic, revising up their expected excess returns.

It is worth emphasizing that the source of return extrapolation in the model is clearly not the only one. The belief dynamics specified in (21) are only affected by crashes but are unaffected by small Brownian shocks. It is evident that positive fundamental shocks in economic booms can lead to overoptimism, and the theory proposed in Barberis, Greenwood, Jin, and Shleifer (2014) analyzes the effect of realized Brownian shocks on investors' beliefs and asset prices.

In contrast to the procyclical pattern for the perceived expected excess return, the true expected excess return exhibits a strong countercyclical pattern; the correlation between the two time series is -0.60. It is exactly when market is filled with fear after crashes that the true expected excess return becomes large—to some extent, loss of confidence is why the true expected excess return, as well as the true Sharpe ratio, is high during busts. The negative correlation between the perceived and the true expected excess returns is consistent with the finding of Amromin and Sharpe (2013) and Greenwood and Shleifer (2014) that investors' expectations of returns are negatively correlated with realized returns.

The time-series results in Figures 3a and 3b are based on one particular simulated dividend path. To complete the discussion of the model's time-series implications, I present in Figure 4 several quantities of interest, with each quantity averaged over 1000 simulated paths, and with specified initial states and timing of subsequent crash realizations. Specifically, I compute the average amount of credit expansion, measured by changes in the level of debt over either one year or two years in the absence of an intervening crash event, for various initial values of x_0 and λ_0. I then compute the average severity of the crash in the asset price assuming that a dividend crash hits the market by the end of the one-year or two-year period. In this way, I examine the effect that the initial belief, the initial wealth, and the length of the expansion period have on the relation between credit expansion and crash severity. I further examine the consequence of the crash by computing the average recovery time in subsequent periods.[23]

[**Place Figure 4 about here**]

During a period without a crash, Figure 4 suggests that the amount of credit expansion is hump-shaped in speculators' initial wealth: with a low wealth level, the amount of debt speculators can borrow is limited by their wealth; with a medium wealth level, speculators rely heavily on debt to invest in the risky asset as their optimism increases over time, resulting in a large credit expansion; with a high wealth level, a general equilibrium effect pushes down the expected excess return so that speculators take on less debt. The end-of-period crash severity in the asset price, however, is

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[23]The recovery time is defined as the amount of time the economy spends without borrowing and lending between speculators and banks. I look at the subsequent three years following a crash event; if the credit market stops operating for more than three years, the recovery time is capped at three.

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monotonically increasing in speculators' initial wealth level since the higher speculator wealth leads to a larger impact of speculators' beliefs on the asset price, whether or not they are levered. The post-crash recovery time is also monotonically increasing in speculators' wealth because a larger crash requires more time to correct.

If one views the speculators in the model as households, a group that does not have enough wealth to absorb the entire supply of the risky asset, then the low and medium wealth levels are of more empirical relevance. In this range, higher speculator wealth causes a bigger credit expansion, a more severe crash, and a longer recovery time. In other words, higher speculator wealth strengthens the relation between credit expansion and the severity of subsequent crashes.

Given a low initial wealth level for speculators, the size of credit expansion is hump-shaped in the initial belief λ_0: a medium initial level of λ_0 tends to be followed by the largest increase in the level of optimism, resulting in a bigger size of credit expansion. Given a medium-to-high initial wealth level, however, the amount of credit expansion becomes monotonically decreasing in λ_0: a lower λ_0 helps speculators to accumulate wealth at a faster rate during the early stage of the credit expansion and this amplifies their need for debt as they becomes more optimistic over time; the same intuition also explains why the end-of-period crash severity is larger for speculators with lower λ_0. The recovery time is increasing in λ_0 since, with higher λ_0, speculators become more pessimistic after the crash so it takes a longer time for borrowing to resume.

Figure 4 indicates that a longer expansion period naturally leads to a larger amount of credit expansion and a larger crash severity. Interestingly, it also shows that a longer expansion period leads to a shorter rather than longer recovery time in the model. This result comes from the fact that a longer credit expansion increases the level of optimism to a larger extent and therefore results in less pessimistic beliefs after a crash event.

Up to this point, I have been assuming that only speculators have incorrect beliefs about crash likelihood and both speculators and banks have fully correct beliefs about crash severity. In such a model, banks do not take any losses after crashes since they have fully thought through all the consequences of a crash event and have requested a sufficient amount of collateral to hedge the crash risk. Nevertheless, a growing empirical literature suggests that banks do take significant losses during financial crises, and that, to some degree, they are caught by surprise: Foote et al. (2012) argue that banks were among the biggest losers in the recent Great Recession; Baron and Xiong (2014) show that the mean excess return for the bank equity index following credit expansions is substantially negative; Cheng, Raina, and Xiong (2013) provide evidence that mid-level managers in securitized finance were significantly more likely to purchase second homes for their personal accounts compared to some control groups during the housing bubble and that the performance of their home portfolios was much worse; Coval et al. (2014) demonstrate that neglected crash risk by suppliers of downside economic insurance products leads to large unexpected losses. Given these findings, it is important to consider the case where agents have incorrect beliefs about crash severity. I do this in Section IV.

IV. Incorrect Beliefs about Crash Severity

In this section, I further generalize the model to allow for incorrect beliefs about the severity of the price crash. To model incorrect beliefs about crash severity and provide a micro-foundation for them, I take a bounded-rationality approach.[24] Recall from Eq. (43) that speculators and banks need to fully anticipate the deleveraging taken by *all* the speculators in the economy to make a rational forecast of the size of a potential crash. Such complete awareness may be implausible. If a single-risky-asset framework is a reduced-form model for multiple assets, then assessing deleveraging for all assets requires, at the very least, an accurate estimate of systematic risk. Investors may make mistakes in this assessment—indeed, Coval et al. (2009) show that investors underestimated both the probability of mortgage defaults and the correlations among these defaults during the most recent credit boom. It is therefore reasonable to assume that, ex ante, speculators and banks may have limited capacity to assess the post-crash market-wide deleveraging. To capture this form of bounded rationality, I modify (43) as

$$\frac{1 - \hat{\kappa}_{P,t}^S(x_t, \lambda_t)}{1 - \kappa} l(x_t, \lambda_t) = l\left(\underbrace{\left[\xi + (1 - \xi)\frac{1 - w_t\hat{\kappa}_{P,t}^S(x_t, \lambda_t)}{1 - \kappa}\right]}_{\substack{\text{wealth effect and belief effect} \\ \text{with bounded rationality}}} x_t, \underbrace{\lambda_t + b(\lambda_t)}_{\text{belief effect}} \right). \tag{44}$$

Eq. (44) specifies the ex ante crash severity *perceived* by speculators $\hat{\kappa}_{P,t}^S$ given the current state variables x_t and λ_t.[25] The parameter ξ measures the degree of bounded rationality. When $\xi = 1$, investors are completely myopic in assessing the impact of a crash on the state variable x_t; they ignore the effect of deleveraging by the other investors and believe that the reduction in others' wealth in a crash will have the same percentage size as the reduction in the dividend. When $\xi = 0$, (44) reduces to (43) and speculators have completely rational forecasts about the severity of a price crash. For simplicity, I assume that banks and speculators are equally irrational in assessing the crash severity. In other words, $\hat{\kappa}_{P,t}^S = \hat{\kappa}_{P,t}^B = \hat{\kappa}_{P,t}$.

Two observations about the modelling assumption on the incorrect beliefs about crash severity are worth mentioning. First, (44) only determines the perceived crash severity, not the true or realized crash severity $\kappa_{P,t}$. When $\hat{\kappa}_{P,t}^B < \kappa_{P,t}$, the collateral requirement may be insufficient to repay the loan, and banks could suffer losses upon a crash. Second, (44) can generate either underestimation or overestimation of the crash severity depending on the state of the world; the

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[24]Alternatively, incorrect beliefs about crash severity can also be motivated in the same way as with incorrect beliefs about the crash likelihood, namely, using the availability heuristic. After a sequence of crashes, such events become very salient and easy to recall; as a result, investors may overestimate the magnitude of future crashes. Conversely, after a prolonged absence of crashes, investors may find it difficult to retrieve crash-related information from memory, and therefore underestimate the severity of future crashes.

[25]Note that, in principle, speculators and banks could also hold incorrect beliefs about the size of the dividend crash, κ. However, given that κ is a constant and can be learned after just a single crash observation, I assume that it is known to all agents in the economy.

size of misestimation is tied to the value of the parameter ξ.

Given incorrect beliefs about crash severity, the state-dependent expected capital gain, as perceived by speculators, now includes a severity term in addition to the intensity effect introduced earlier

$$\hat{g}_{P,t} = g_{P,t} + \underbrace{\kappa_{P,t}(\lambda - \lambda_t)}_{\text{intensity effect}} + \underbrace{\lambda_t(\kappa_{P,t} - \hat{\kappa}_{P,t})}_{\text{severity effect}}. \tag{45}$$

Underestimating the crash severity leads speculators to overestimate the expected capital gain, and conversely, overestimating the crash severity leads speculators to underestimate the expected capital gain.

By (45), the perceived evolution of the equilibrium price is

$$\begin{aligned} dP_t/P_t &\mathrel{\hat{=}} (\hat{g}_{P,t} + \lambda_t\hat{\kappa}_{P,t})dt + \sigma_{P,t}d\omega_t - \hat{\kappa}_{P,t}dN_t \\ &\mathrel{\hat{=}} (g_{P,t} + \lambda\kappa_{P,t})dt + \sigma_{P,t}d\omega_t - \hat{\kappa}_{P,t}dN_t. \end{aligned} \tag{46}$$

The symbol "$\hat{=}$" indicates that the equation holds only in speculators' perception. Recall that, as discussed in (26), $\hat{g}_{P,t} + \lambda_t\hat{\kappa}_{P,t}$ and $g_{P,t} + \lambda\kappa_{P,t}$ have been assumed to be equal to focus attention on errors in beliefs about crash risk. This means that the perceived price evolution matches the true evolution except when a crash occurs.

Given (46), speculator i perceives that her wealth evolution is

$$\begin{aligned} dW_t^i &\mathrel{\hat{=}} -C_t^i dt + rW_t^i dt + w_t^i W_t^i[(\hat{g}_{P,t} + \lambda_t\hat{\kappa}_{P,t} + D_t/P_t - r)dt + \sigma_{P,t}d\omega_t - \hat{\kappa}_{P,t}dN_t] \\ &\mathrel{\hat{=}} -C_t^i dt + rW_t^i dt + w_t^i W_t^i[(g_{P,t} + \lambda\kappa_{P,t} + D_t/P_t - r)dt + \sigma_{P,t}d\omega_t - \hat{\kappa}_{P,t}dN_t]. \end{aligned} \tag{47}$$

with $w_t^i \le \hat{\kappa}_{P,t}^{-1}$.

Introducing incorrect beliefs about the crash severity does not bring in new state variables for the economy. As before, $P_t/D_t = l(x_t, \lambda_t)$.

The definition of an equilibrium in this economy is given below.

Definition 3. *An equilibrium is characterized by the price process $\{P_t\}$ and the consumption and portfolio decisions $\{C_t^i, w_t^i\}$ for each speculator i such that:*

1) Given the price process $\{P_t\}$ and the constant interest rate r, speculator i's consumption and portfolio decisions solve

$$\max_{\{C_t^i, w_t^i \le \hat{\kappa}_{P,t}^{-1}\}_{t\ge 0}} \mathbb{E}_0^i\left[\int_0^\infty e^{-\rho t}\ell n(C_t^i)dt\right], \tag{48}$$

subject to her belief evolution in (21) and her perceived wealth evolution in (47);

2) On a per-capita basis, the market clearing condition for the risky asset

$$\mu W_t w_t + (1 - \mu)Q_t = P_t \tag{49}$$

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is satisfied at each point in time.

In Lemma 3, I characterize speculators' optimal consumption and portfolio decisions.

Lemma 3. *For the model with incorrect beliefs about both the crash likelihood and crash severity, speculator i's optimal consumption stream is*

$$C_t^i = \rho W_t^i, \tag{50}$$

and her optimal risky asset investment w_t^i is the lower root of

$$\hat{g}_{P,t} + \lambda_t \hat{\kappa}_{P,t} + l^{-1} - r - w_t^i \sigma_{P,t}^2 - \frac{\lambda_t \hat{\kappa}_{P,t}}{1 - w_t^i \hat{\kappa}_{P,t}} = 0. \tag{51}$$

That is,

$$(w_t^i)^* = w_t = \frac{\hat{\kappa}_{P,t} A_t + \sigma_{P,t}^2 - \sqrt{(\hat{\kappa}_{P,t} A_t - \sigma_{P,t}^2)^2 + 4\lambda_t \hat{\kappa}_{P,t}^2 \sigma_{P,t}^2}}{2\hat{\kappa}_{P,t}\sigma_{P,t}^2}, \tag{52}$$

where $A_t \equiv \hat{g}_{P,t} + \lambda_t \hat{\kappa}_{P,t} + l^{-1} - r = g_{P,t} + \lambda\kappa_{P,t} + l^{-1} - r$ is the average excess return of the risky asset in the absence of a crash.

Proof. See Appendix A.3. ∎

In the proposition below, I present the analytical results that describe the asset pricing implications of the model.

Proposition 3. *For the model with incorrect beliefs about both the crash likelihood and crash severity, the Brownian volatility, the average growth rate of the risky asset in the absence of a crash, and the crash severity on the asset price are, respectively,*

$$\sigma_{P,t}(x_t, \lambda_t) = \frac{1 - (l_x/l)x_t}{1 - (l_x/l)(c_0 l - c_1)}\sigma_D, \tag{53}$$

$$(g_{P,t} + \lambda\kappa_{P,t})(x_t, \lambda_t) \equiv \bar{g}_{P,t}(x_t, \lambda_t) = [1 - (l_x/l)(c_0 l - c_1)]^{-1}\left\{g_D + \lambda\kappa + \sigma_D\sigma_{P,t} - \sigma_D^2\right.$$

$$+ (l_x/l)x_t[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + (l^{-1} - r - \sigma_D\sigma_{P,t})(c_0 l - c_1)/x_t] \tag{54}$$

$$\left. + (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)[(l/l_x)(\sigma_{P,t} - \sigma_D)]^2\right\},$$

$$\hat{\kappa}_{P,t}(x_t, \lambda_t) = \frac{\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t}{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t}. \tag{55}$$

The fraction of speculators' wealth invested in the risky asset is

$$w_t(x_t, \lambda_t) = (c_0 l - c_1)/x_t. \tag{56}$$

The price-dividend ratio $l(x_t, \lambda_t)$ is the solution to

$$
\frac{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t]((c_0 l - c_1)/x_t - 1)}{(1 - \kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}} l(x_t, \lambda_t)
$$
$$
= l\left(\left[\xi + \frac{(1 - \xi)\lambda_t}{(1 - \kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}\right] x_t, \lambda_t + b(\lambda_t)\right),
$$
(57)

a second-order partial differential-difference equation after substitution of (53) through (55), with boundary conditions

$$
\lim_{x_t \to \infty} l(x_t, \lambda_t) = [r - \lim_{x_t \to \infty} \hat{g}_{P,t}(x_t, \lambda_t)]^{-1} \equiv m(\lambda_t),
$$
(58)

$$
\lim_{x_t \to 0} l(x_t, \lambda_t) = c_1/c_0, \qquad \lim_{x_t \to 0} l_x(x_t, \lambda_t) = w(0, \lambda_t)/c_0,
$$
(59)

for $\forall \lambda_t \in (\lambda_m, \lambda_h)$, and the function $m(\lambda_t)$ is the solution to

$$
m'(\lambda_t)(a(\lambda_t) - \lambda_t b(\lambda_t)) = m(\lambda_t)\left(\lambda_t - g_D - \lambda\kappa - m^{-1}(\lambda_t) + r - \frac{(1 - \kappa)\lambda_t m(\lambda_t + b(\lambda_t))}{m(\lambda_t)}\right).
$$
(60)

The true crash severity $\kappa_{P,t}$ is the solution to

$$
\frac{1 - \kappa_{P,t}}{1 - \kappa} l(x_t, \lambda_t) = \begin{cases} l\left(\dfrac{1 - w_t \kappa_{P,t}}{1 - \kappa} x_t, \lambda_t + b(\lambda_t)\right) & w_t \leq 1 \\ l\left(\dfrac{1 - w_t \kappa_{P,t} + (w_t - 1)\max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})]}{1 - \kappa} x_t, \lambda_t + b(\lambda_t)\right) & w_t > 1 \end{cases}.
$$
(61)

The arguments, sometimes omitted, of the function l and of its derivatives in Eqs. (53) to (57) are x_t and λ_t, except on the right-hand side of (57) where they are the post-crash values.

The unanticipated percentage loss for speculators, defined as the difference between their realized dollar loss upon a crash and their anticipated dollar loss divided by their pre-crash wealth level, is

$$
UPL^S(x_t, \lambda_t) = \begin{cases} w_t(\kappa_{P,t} - \hat{\kappa}_{P,t}) & w_t \leq 1 \\ w_t(\kappa_{P,t} - \hat{\kappa}_{P,t}) - (w_t - 1)\max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})] & w_t > 1 \end{cases}.
$$
(62)

The unanticipated total loss for speculators, defined as their unanticipated dollar loss normalized by the dividend level, and denoted UTL^S, is $x_t UPL^S$.

The unanticipated percentage loss for banks, defined as their realized dollar loss upon a crash divided by the pre-crash dollar amount of collateralized funding they provide, is

$$
UPL^B(x_t, \lambda_t) = \begin{cases} 0 & w_t \leq 1 \\ \max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})] & w_t > 1 \end{cases}.
$$
(63)

The unanticipated total loss for banks, defined as their unanticipated dollar loss normalized by the dividend level, and denoted UTL^B, is $(w_t - 1)x_t UTL^B$.

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Proof. See Appendix A.3.   ∎

A few remarks regarding the analytical results in Lemma 3 and Proposition 3 are worth making. First, all the asset pricing implications, except the realized crash severity, are completely determined by the perceptions of speculators and banks. Given the incorrect beliefs about the crash severity, speculators and banks will be surprised by a crash in that its magnitude will differ from what they expected. Second, when speculators and banks underestimate the crash severity and when speculators take on leverage, they both experience unexpected losses when a crash occurs. The loss size is linked to the amount of collateralized funding and to banks' ability to seize speculators' personal wealth. When $w_t \leq 1$, speculators do not borrow; they takes the entire unexpected loss. When $w_t > 1$, they do borrow; the unexpected loss is shared between banks and speculators, with the banks' share being smaller, the greater their ability to seize speculators' other assets. Lastly, the boundary conditions in Eqs. (58), (59), and (60) are identical to those in Proposition 2—bounded rationality plays no role in the two limiting cases. When $x_t \to \infty$, speculators invest most of their wealth in the safe asset and hence experience no deleveraging upon a crash. When $x_t \to 0$, the risky asset is mostly held by the long-term investors so speculators and banks do not affect the asset price.

To quantify the implications of incorrect beliefs about crash severity on the asset price and the trading behavior, I use numerical methods to solve (57) and (61). As in Section III, I define $z_t \equiv (x_t - \gamma)/(x_t + \gamma)$ and $h(z_t, \lambda_t) \equiv l(x(z_t), \lambda_t)$. The numerical procedure is outlined in Appendix B.3.

[Place Figure 5a and Figure 5b about here]

Figures 5a and 5b provide 3-D and 2-D views of the model implications, respectively. I have already discussed the implications of incorrect beliefs about crash likelihood, so I focus here on the additional implications yielded by incorrect beliefs about crash severity.

Figures 5a and 5b show that the realized and the perceived crash severities can be quite different in the presence of bounded rationality. For most values of (x_t, λ_t), the realized crash severity is greater than the perceived crash severity. The ratio of the realized over the perceived crash severity reaches its highest value when speculators' expected crash intensity λ_t is low and when their wealth level x_t is intermediate. With a low λ_t and an intermediate x_t, speculators are optimistic and borrow heavily from banks but then need to delever after a crash. As each speculator neglects market-wide deleveraging, they substantially underestimate the crash severity—for example, when $\lambda_t = \lambda_m$ and $x_t = 19.0$, speculators underestimate the crash severity by 22%. For some values of (x_t, λ_t), the crash severity can be overestimated. The ratio reaches its lowest value when λ_t is high and when x_t is intermediate. With a high λ_t and an intermediate x_t, speculators are pessimistic and only invest a small fraction of their wealth in the risky asset and invest more after a crash occurs to rebalance their portfolios. As each speculator neglects this market-wide rebalancing, she overestimates the crash severity—for example, when $\lambda_t = \lambda_h$ and $x_t = 54.8$, speculators overestimate the crash severity by 7%. These results show that the bounded-rationality approach can generate underestimation and overestimation of the crash severity in different states of the world.

For speculators, incorrect beliefs about crash severity can lead to unanticipated losses in crashes. Both the unanticipated percentage loss and the unanticipated total loss are largest when speculators' perceived crash likelihood λ_t is low, but not too low, and when their wealth level x_t is intermediate. With a low λ_t and an intermediate x_t, speculators tend to underestimate the crash severity and hence experience a larger-than-expected loss when a crash occurs. However, when λ_t is too close to its minimum level, the true crash severity is smaller because speculators retain much of their optimism after a crash, resulting in a smaller unanticipated loss.

Banks, on the other hand, do not expect any losses upon crashes. However, with incorrect beliefs about crash severity, they will sometimes experience them. The largest unanticipated percentage or total loss for banks occurs when banks provide a lot of funding but significantly overestimate the post-crash collateral value of the underlying asset. Once again, this happens when λ_t is low, but not too low, and when x_t is intermediate—for example, when $\lambda_t = 6.25\%$ and $x_t = 29.0$, banks lose 5% on each dollar they provide. When speculators' risky asset portfolio weight w_t is less than one, banks provide no funding to speculators and therefore experience no losses when a crash occurs.

As the model is dynamic, I can also examine the time-series implications of incorrect beliefs about crash severity. I look at the size of crash expansions, the crash severity, and bank losses, with each quantity averaged over 1000 simulated dividend paths. Specifically, I first compute the average size of credit expansion over either one year or two years in the absence of an intervening crash event for various initial values of x_0 and λ_0 and with the bounded rationality parameter ξ equal to 0.5 or 1. I then compute the average crash severity and the average unanticipated percent loss for banks, assuming that a crash occurs by the end of this one-year or two-year period. The results are summarized in Figure 6.

[**Place Figure 6 about here**]

Figure 6 suggests that incorrect beliefs about crash severity have a much smaller impact on the size of the credit expansion relative to incorrect beliefs about crash likelihood. When the perceived crash likelihood is low, the overall perceived crash risk is low and insensitive to changes in the perceived crash severity. On the other hand, incorrect beliefs about crash severity have strong implications for the losses taken by banks in crashes. Three specific observations are worth noting. First, a stronger degree of bounded rationality leads to a larger underestimation of crash severity, which, in turn, leads to larger post-crash bank losses—for example, when $x_0 = 20$ and $\lambda_0 = 0.4$, increasing ξ from 0.5 to 1 on average almost double banks' unanticipated percentage loss from 1.66% to 3.13% if a crash occurs after one year. Second, the impact of incorrect beliefs about crash severity on bank losses is significantly affected by incorrect beliefs about crash likelihood— for example, when $x_0 = 20$ and $\xi = 1$, decreasing λ_0 from 0.6 to 0.4 on average increases banks' unanticipated percentage loss by 64%, from 1.91% to 3.13%, if a crash occurs after one year. With a lower λ_0, speculators are more optimistic at the end of the expansion period, and their leverage is therefore higher when the crash occurs; with higher leverage, the deleveraging effect is stronger and therefore bounded rationality about market-wide deleveraging generates a more severe loss

for banks and speculators. Finally, a longer period of expansion amplifies the effect of incorrect beliefs about crash severity on bank losses. After a long "quiet" period, speculative capital grows significantly with the help of collateralized funding. If a crash occurs in such situations, the amount of speculative capital associated with the post-crash pessimism remains large, resulting in a bigger drop in the collateral value. Given this, underestimation of the crash severity can be quite costly to banks. A longer period of credit expansion leads to more financial fragility in the model.

Note that bounded rationality in assessing market-wide deleveraging is just *one* possible source of incorrect beliefs about crash severity; there may be many other sources. The compounding impact of incorrect beliefs about crash severity on losses for banks and speculators can be much larger than estimated in the model.

V.   Conclusion

In this paper, I have developed a dynamic equilibrium model that studies the impact of incorrect beliefs about crash risk on asset prices, portfolio decisions, and bank losses. The model generates a strong amplification—the magnitude of a price crash can be much larger than the magnitude of the crash in fundamentals when financially constrained speculators underestimate crash likelihood and take on excessive leverage and when their post-crash beliefs about the likelihood of future crashes differ a lot from their pre-crash beliefs. The standard wealth effect and the belief dynamics about crash risk together give rise to a positive relation between credit expansion and the severity of a crash in the asset price in subsequent periods—financial fragility builds up over time as the perceived crash risk decays, and a credit boom arises endogenously along the way with an increasing amount of crash risk borne by both speculators and banks. With a lower perceived crash likelihood following a prolonged credit expansion, the model predicts that average excess returns of the risky asset become lower even when the financial system becomes more fragile. In addition, bounded rationality in assessing the post-crash scenario of market-wide deleveraging can lead to speculator defaults and bank losses.

The model shows that the impact on beliefs of any policy intervention that attempts to reduce financial fragility cannot be ignored when evaluating its effectiveness. For example, when investors are overly optimistic and have taken on high leverage, the effects of an intervention that forces them to reduce leverage will also depend on its impact on investor confidence. If investors interpret the intervention to mean an increased crash likelihood, then the deterioration in beliefs might be much more severe after a crash. The overall effect of the intervention could then be detrimental. On the other hand, if the policy makers can keep the level of confidence unchanged while reducing leverage, then it will limit the impact of incorrect beliefs and reduce financial fragility. A more careful analysis of the model's policy and welfare implications is left for future research.

The model can be further extended in a number of ways. In my setting, although the size of a crash event is endogenous, its origin is still exogenous—it occurs whenever a Poisson event arrives. Endogenizing such an event requires developing more complex belief dynamics about crash

intensity that are affected not only by past occurrences of price crashes but also by the sizes of those crashes. In such extensions, it is conceivable that belief dynamics about crash risk would interact with market liquidity and funding liquidity, as analyzed in Brunnermeier and Pedersen (2009), and create additional amplification, both contemporaneously and intertemporally. For example, when funding liquidity is tightened, speculators will reduce their trading positions and push down asset prices; lower asset prices then may lead to larger perceived crash risk by both speculators and banks and result in higher margin requirements, which in turn may trigger further deleveraging and deterioration in beliefs, and so on.

The current framework only includes one risky asset and can be extended to allow for multiple assets. Kyle and Xiong (2001) show that the time-varying wealth of convergence traders can create contagion across assets. In a multi-asset extension of my model, both the time-varying speculative capital and the dynamic beliefs about crash risk would likely contribute to contagion. If a crash hits the asset that serves as collateral for investing in other risky assets, then the deterioration in beliefs on this particular asset would lead to a drop in its collateral value, making it difficult for speculators to invest in other risky assets, and a large reduction in wealth caused by deleveraging would force speculators to cut back further on investments in all risky assets, triggering additional price drops. In such a framework, further allowing for both positive and negative Poisson shocks of equal size on the dividend process may endogenously generate negatively skewed market returns.

Appendices

A. Analytical Results for the Equilibrium

1. Solving the Equilibrium for the Rational Benchmark

Proof of Lemma 1. First, define the value function for speculator i as

$$J(W_t^i, x_t) \equiv \max_{\{C_{t+s}^i, w_{t+s}^i \leq \kappa_{P,t+s}^{-1}\}_{s\geq 0}} \mathbb{E}_t^i \left[\int_0^\infty e^{-\rho t} \ell n(C_{t+s}^i) ds \right], \tag{A.1}$$

and write the evolution of the state variable x_t as

$$dx_t = g_{x,t}(x_t)dt + \sigma_{x,t}(x_t)d\omega_t - \kappa_{x,t}(x_t)(dN_t - \lambda dt). \tag{A.2}$$

I verify the Markov nature of x_t later in the proof of Proposition 1.

Given the wealth evolution in (5) and the evolution of x_t in (A.2), I use the stochastic dynamic programming approach developed in Merton (1971) to derive the Hamilton-Jacobi-Bellman (HJB) equation for speculator i as

$$\rho J(W_t^i, x_t) = \max_{\{C_t^i, w_t^i \leq \kappa_{P,t}^{-1}\}} \left\{ \begin{array}{l} \ell n(C_t^i) + J_W W_t^i[-C_t^i/W_t^i + r + w_t^i(g_{P,t} + \lambda \kappa_{P,t} + l^{-1}(x_t) - r)] \\ +\frac{1}{2} J_{WW}(W_t^i w_t^i \sigma_{P,t})^2 + J_x(g_{x,t} + \lambda \kappa_{x,t}) + \frac{1}{2} J_{xx}\sigma_{x,t}^2 + J_{Wx} w_t^i W_t^i \sigma_{P,t} \sigma_{x,t} \\ +\lambda\{J((1 - w_t^i \kappa_{P,t})W_t^i, (1 - \kappa)^{-1}(1 - w_t \kappa_{P,t})x_t) - J(W_t^i, x_t)\} \end{array} \right\}. \tag{A.3}$$

Upon the occurrence of a Poisson event, the state variable x_t is scaled down with the reduction of the per-capita wealth by a factor of $(1 - w_t \kappa_{P,t})$, where w_t is speculators' per-capita investment in the risky asset. It is the aggregate investment that determines the reduction in per-capita wealth so w_t is not a choice variable for speculator i.

Now conjecture that the value function has the form

$$J(W_t^i, x_t) = \rho^{-1} \ell n(W_t^i) + j(x_t). \tag{A.4}$$

The first order condition of (A.3) with respect to C_t^i, together with the conjecture in (A.4), yields

$$C_t^i = J_W^{-1} = \rho W_t^i. \tag{A.5}$$

As is standard for log-utility preferences, the consumption rate is proportional to wealth with a constant of proportionality ρ.

Substituting (A.4) and (A.5) back to the HJB equation gives

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$$\rho j = \max_{w_t^i \leq \kappa_{P,t}^{-1}} \left\{ \begin{array}{l} \ell n(\rho) + \rho^{-1}[r - \rho + w_t^i(g_{P,t} + \lambda\kappa_{P,t} + l^{-1}(x_t) - r)] \\[2mm] -\frac{1}{2}\rho^{-1}(w_t^i)^2\sigma_{P,t}^2 + (g_{x,t} + \lambda\kappa_{x,t})j'(x_t) + \frac{1}{2}\sigma_{x,t}^2 j''(x_t) \\[2mm] +\lambda\{\rho^{-1}\ell n(1 - w_t^i\kappa_{P,t}) + j((1 - \kappa)^{-1}(1 - w_t\kappa_{P,t})x_t) - j(x_t)\} \end{array} \right\}, \tag{A.6}$$

a reduced-HJB equation for the j function. The absence of W_t^i in (A.4) verifies the conjecture in (A.4). The first order condition with respect to w_t^i gives

$$g_{P,t} + \lambda\kappa_{P,t} + l^{-1} - r - w_t^i\sigma_{P,t}^2 - \frac{\lambda\kappa_{P,t}}{1 - w_t^i\kappa_{P,t}} = 0, \qquad w_t^i \leq \kappa_{P,t}^{-1}, \tag{A.7}$$

with the last term capturing the effect of the Poisson shock on investors' portfolio decisions. Eq. (A.7) is a quadratic equation for w_t^i with one root above $\kappa_{P,t}^{-1}$ and one root below $\kappa_{P,t}^{-1}$. The leverage constraint excludes the larger root so the optimal holding is

$$(w_t^i)^* = w_t = \frac{\kappa_{P,t}A_t + \sigma_{P,t}^2 - \sqrt{(\kappa_{P,t}A_t - \sigma_{P,t}^2)^2 + 4\lambda\kappa_{P,t}^2\sigma_{P,t}^2}}{2\kappa_{P,t}\sigma_{P,t}^2}, \tag{A.8}$$

where $A_t \equiv g_{P,t} + \lambda\kappa_{P,t} + l^{-1} - r$.[26] ∎

Proof of Proposition 1. The first step is to write the evolution of the state variable x_t in a more fundamental way. Given the optimal consumption plan in (A.5), the optimal per-capita wealth of the speculators evolves as

$$dW_t/W_t = (r - \rho + w_t A_t)dt + w_t\sigma_{P,t}d\omega_t - w_t\kappa_{P,t}dN_t. \tag{A.9}$$

Applying Ito's lemma to both sides of $x_t = W_t/D_t$ and using Eqs. (1) and (A.9) gives

$$\begin{aligned} dx_t/x_t = {}&(r - \rho + w_t A_t - g_D - \lambda\kappa - w_t\sigma_D\sigma_{P,t} + \sigma_D^2)dt \\ &+ (w_t\sigma_{P,t} - \sigma_D)d\omega_t + (\kappa - w_t\kappa_{P,t})(1 - \kappa)^{-1}dN_t. \end{aligned} \tag{A.10}$$

Comparing (A.10) with (A.2) leads to

$$g_{x,t} + \lambda\kappa_{x,t} = (r - \rho + w_t A_t - g_D - \lambda\kappa - w_t\sigma_D\sigma_{P,t} + \sigma_D^2)x_t, \tag{A.11}$$

$$\sigma_{x,t} = (w_t\sigma_{P,t} - \sigma_D)x_t, \qquad \kappa_{x,t} = (w_t\kappa_{P,t} - \kappa)(1 - \kappa)^{-1}x_t.$$

---

[26]The second derivative of the reduced-HJB equation in (A.6) with respect to w_t^i is negative when $w_t^i \in (0, \kappa_{P,t}^{-1})$, suggesting that the solution in (A.8) indeed maximizes rather than minimizes the value function.

Next, applying Ito's lemma to both sides of $P_t/D_t = l(x_t)$ and using Eqs. (1), (2), and (A.2) yields

$$g_{P,t} + \lambda\kappa_{P,t} - g_D - \lambda\kappa + \sigma_D^2 - \sigma_D\sigma_{P,t} = (l'/l)(g_{x,t} + \lambda\kappa_{x,t}) + \tfrac{1}{2}(l''/l)\sigma_{x,t}^2,$$

$$\sigma_{P,t} - \sigma_D = (l'/l)\sigma_{x,t}, \tag{A.12}$$

$$(1-\kappa)^{-1}(1-\kappa_{P,t})l(x_t) = l((1-\kappa)^{-1}(1-w_t\kappa_{P,t})x_t).$$

The argument of function l and its derivatives is sometimes omitted for brevity.

Substituting the demand function of (3) for the long-term investors into the market clearing condition in (6) yields

$$w_t(x_t) = \frac{(k+1-\mu)(r-g_D)l(x_t) - (1-\mu)}{\mu k(r-g_D)x_t} \equiv (c_0 l(x_t) - c_1)x_t^{-1}, \quad \forall t. \tag{A.13}$$

Combining (A.11), (A.12), and (A.13) allows me to derive the Brownian volatility and the average growth rate of the risky asset in the absence of a crash as

$$\sigma_{P,t} = \frac{1 - (l'/l)x_t}{1 - (l'/l)w_t x_t}\sigma_D = \frac{1 - (l'/l)x_t}{1 - (l'/l)(c_0 l - c_1)}\sigma_D, \tag{A.14}$$

$$\begin{aligned}
g_{P,t} + \lambda\kappa_{P,t} \equiv \bar{g}_{P,t} &= [1 - (l'/l)w_t x_t]^{-1}\left\{g_D + \lambda\kappa + \sigma_D\sigma_{P,t} - \sigma_D^2\right. \\
&\quad + (l'/l)x_t[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + w_t(l^{-1} - r - \sigma_D\sigma_{P,t})] \\
&\quad \left. + \tfrac{1}{2}(l''/l)\sigma_{x,t}^2\right\} \\
&= [1 - (l'/l)(c_0 l - c_1)]^{-1}\left\{g_D + \lambda\kappa + \sigma_D\sigma_{P,t} - \sigma_D^2\right. \\
&\quad + (l'/l)x_t[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + (l^{-1} - r - \sigma_D\sigma_{P,t})(c_0 l - c_1)/x_t] \\
&\quad \left. + \tfrac{1}{2}(l''/l)[(l/l')(\sigma_{P,t} - \sigma_D)]^2\right\}.
\end{aligned} \tag{A.15}$$

Substituting (A.13) into the first-order condition in (A.7) gives

$$\kappa_{P,t} = \frac{\bar{g}_{P,t} + l^{-1} - r - w_t^i\sigma_{P,t}^2}{\lambda + (\bar{g}_{P,t} + l^{-1} - r - w_t^i\sigma_{P,t}^2)w_t^i} = \frac{\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t}{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t}. \tag{A.16}$$

Further substituting (A.13) and (A.16) into the last equation in (A.12) gives

$$\begin{aligned}
&\frac{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t]((c_0 l - c_1)/x_t - 1)}{(1-\kappa)\{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}l(x_t) \\
&= l\left(\frac{\lambda}{(1-\kappa)\{\lambda + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}x_t\right),
\end{aligned} \tag{A.17}$$

where $\sigma_{P,t}$ and $\bar{g}_{P,t}$ are determined as functions of l and its derivatives in (A.14) and (A.15). Eq. (A.17) is a second-order differential-difference equation for l.

To fully determine the functional form of l, two boundary conditions need to be imposed. First, consider the limiting case as $x_t = W_t/D_t \to \infty$. In this case, (A.13) implies that $w_t \to 0$, and the

first-order condition in (A.7) then implies that the conditional expected excess return must converge to zero. That is, as the fraction of speculators' wealth invested in the risky asset approaches zero so does the risk, and speculators' behavior approaches risk-neutrality. Under these properties, a proportional economy forms, and from (A.7)

$$\lim_{x_t \to \infty} l(x_t) = (r - g_D)^{-1}. \tag{A.18}$$

In this limiting case, the Gordon growth model applies, and the long-term investors' dollar demand reduces to zero.

Second, consider the case as $x_t \to 0$. In this case, speculator wealth goes to zero, and the long-term investors must hold the entire supply of the risky asset. From the demand function of the long-term investors in (3) and from the definitions of c_0 and c_1 in (A.13)

$$\lim_{x_t \to 0} l(x_t) = (r - g_D)^{-1}[1 + k/(1 - \mu)]^{-1} = c_1/c_0. \tag{A.19}$$

By (A.8),

$$w(0) \equiv \frac{\kappa A + \sigma_D^2 - \sqrt{(\kappa A - \sigma_D^2)^2 + 4\lambda\kappa^2\sigma_D^2}}{2\kappa\sigma_D^2}, \tag{A.20}$$

where $A \equiv g_D + \lambda\kappa + (r - g_D)(1 + k/(1 - \mu)) - r$. Furthermore, from (A.13) and (A.20)

$$\lim_{x_t \to 0} l'(x_t) = w(0)/c_0. \tag{A.21}$$

Eqs. (A.14) to (A.16) verify the conjecture that x_t is the only state variable that governs the evolutions of $g_{P,t}$, $\sigma_{P,t}$ and $\kappa_{P,t}$. Furthermore, this statement, together with Eqs. (A.11) and (A.13), verifies the claim made in the proof of Lemma 1 that x_t is a Markov process in itself. ■

2. Solving the Equilibrium for the General Model in Section III

The Filtering Problem. The evolution of speculators' beliefs π_t has two components. The first comes from the believed Markov switching between the two states λ_h and λ_l. Although the believed switching is random, it is unobserved and has a deterministic effect on changes in π_t

$$\Pr_t^i\{\tilde{\lambda}_t = \lambda_h\}\Pr_t^i\{\tilde{\lambda}_{t+dt} = \lambda_h|\tilde{\lambda}_t = \lambda_h\} + \Pr_t^i\{\tilde{\lambda}_t = \lambda_l\}\Pr_t^i\{\tilde{\lambda}_{t+dt} = \lambda_h|\tilde{\lambda}_t = \lambda_l\} - \pi_t$$
$$= \pi_t(1 - q_h dt) + (1 - \pi_t)q_l dt - \pi_t = [-\pi_t q_h + (1 - \pi_t)q_l]dt. \tag{A.22}$$

The probabilities labeled with a superscript i are the subjective beliefs of speculator i. The second component arises from the new information carried by the observed Poisson jump. By Bayes' rule, π_t changes to $\pi_t\lambda_h/\lambda_t$ conditional on a jump ($dN_t = 1$) and to $\pi_t(1 - \lambda_h dt)/(1 - \lambda_t dt) = \pi_t[1 + (\lambda_t - \lambda_h)dt] + o(dt)$ conditional on no jump ($dN_t = 0$). Putting these two components

together, the evolution of π_t is

$$d\pi_t = [-\pi_t q_h + (1 - \pi_t)q_l]dt + \lambda_t^{-1}\pi_t(\lambda_h - \lambda_t)(dN_t - \lambda_t dt). \tag{A.23}$$

Because $\lambda_t \equiv \pi_t \lambda_h + (1 - \pi_t)\lambda_l$, the evolution of λ_t is

$$d\lambda_t = (\lambda_h - \lambda_l)[-\pi_t q_h + (1 - \pi_t)q_l]dt + \lambda_t^{-1}\pi_t(\lambda_h - \lambda_l)(\lambda_h - \lambda_t)(dN_t - \lambda_t dt). \tag{A.24}$$

Further substituting for $\pi_t = (\lambda_t - \lambda_l)/(\lambda_h - \lambda_l)$ gives

$$d\lambda_t = a(\lambda_t)dt + b(\lambda_t)(dN_t - \lambda_t dt), \tag{A.25}$$

where $a(\lambda_t) \equiv q_l(\lambda_h - \lambda_t) - q_h(\lambda_t - \lambda_l)$ and $b(\lambda_t) \equiv \lambda_t^{-1}(\lambda_h - \lambda_t)(\lambda_t - \lambda_l)$.

Proof of Lemma 2. The value function for speculator i can be defined as

$$J(W_t^i, x_t, \lambda_t) \equiv \max_{\{C_{t+s}^i, w_{t+s}^i \leq \kappa_{P,t+s}^{-1}\}_{s \geq 0}} \mathbb{E}_t^i\left[\int_0^\infty e^{-\rho t}\ell n(C_{t+s}^i)ds\right], \tag{A.26}$$

with the evolution of λ_t described in (21) and the evolution of x_t generically written as

$$\begin{aligned}
dx_t &= (\hat{g}_{x,t} + \lambda_t \kappa_{x,t})(x_t, \lambda_t)dt + \sigma_{x,t}(x_t, \lambda_t)d\omega_t - \kappa_{x,t}(x_t, \lambda_t)dN_t \\
&= (g_{x,t} + \lambda\kappa_{x,t})(x_t, \lambda_t)dt + \sigma_{x,t}(x_t, \lambda_t)d\omega_t - \kappa_{x,t}(x_t, \lambda_t)dN_t,
\end{aligned} \tag{A.27}$$

where $\hat{g}_{x,t}$ is the conditional expected rate of change of the state variable x_t perceived by speculators. Eqs. (A.27) and (A.25) state that x_t and λ_t are jointly Markov. This statement is verified later in the proof of Proposition 2.

Given the belief evolution in (21), the wealth evolution in (28), and the evolution of x_t in (A.27), speculator i's HJB equation is

$$\rho J(W_t^i, x_t, \lambda_t) = \max_{\{C_t^i, w_t^i \leq \kappa_{P,t}^{-1}\}} \left\{ \begin{aligned} &\ell n(C_t^i) + J_W W_t^i[-C_t^i/W_t^i + r + w_t^i(\hat{g}_{P,t} + \lambda_t \kappa_{P,t} + l^{-1}(x_t, \lambda_t) - r)] \\ &+\tfrac{1}{2}J_{WW}(W_t^i w_t^i \sigma_{P,t})^2 + J_x(\hat{g}_{x,t} + \lambda_t \kappa_{x,t}) + \tfrac{1}{2}J_{xx}\sigma_{x,t}^2 \\ &+J_\lambda(a(\lambda_t) - \lambda_t b(\lambda_t)) + J_{Wx}w_t^i W_t^i \sigma_{P,t}\sigma_{x,t} \\ &+\lambda_t\{J((1 - w_t^i\kappa_{P,t})W_t^i, (1 - \kappa)^{-1}(1 - w_t\kappa_{P,t})x_t, \lambda_t + b(\lambda_t)) - J(W_t^i, x_t, \lambda_t)\} \end{aligned} \right\}. \tag{A.28}$$

The arguments, sometimes omitted, of the derivatives of J are W_t, x_t, and λ_t.

Now conjecture that the value function J takes the form

$$J(W_t^i, x_t, \lambda_t) = \rho^{-1}\ell n(W_t^i) + j(x_t, \lambda_t). \tag{A.29}$$

The first-order condition of (A.28) with respect to C_t^i, together with (A.29), gives

$$C_t^i = J_W^{-1} = \rho W_t^i. \tag{A.30}$$

Substituting the conjectured form of J in (A.29) and the optimal consumption in (A.30) back to the HJB equation in (A.28), one can then obtain the reduced-HJB equation for the j function as

$$\rho j = \max_{w_t^i \le \kappa_{P,t}^{-1}} \left\{ \begin{array}{l} \ell n(\rho) + \rho^{-1}[r - \rho + w_t^i(\hat{g}_{P,t} + \lambda_t \kappa_{P,t} + l^{-1}(x_t, \lambda_t) - r)] \\[2mm] -\frac{1}{2}\rho^{-1}(w_t^i)^2 \sigma_{P,t}^2 + j_x(\hat{g}_{x,t} + \lambda_t \kappa_{x,t}) + \frac{1}{2}j_{xx}\sigma_{x,t}^2 + j_\lambda(a(\lambda_t) - \lambda_t b(\lambda_t)) \\[2mm] +\lambda_t\{\rho^{-1}\ell n(1 - w_t^i \kappa_{P,t}) + j((1-\kappa)^{-1}(1 - w_t \kappa_{P,t})x_t, \lambda_t + b(\lambda_t)) - j(x_t, \lambda_t)\} \end{array} \right\}. \tag{A.31}$$

The conjectured form of J in (A.29) is now verified as (A.31) is independent of W_t^i. The first-order condition of (A.31) with respect to w_t^i gives

$$\hat{g}_{P,t} + \lambda_t \kappa_{P,t} + l^{-1} - r - w_t^i \sigma_{P,t}^2 - \frac{\lambda_t \kappa_{P,t}}{1 - w_t^i \kappa_{P,t}} = 0. \tag{A.32}$$

Noting from (26) that $\hat{g}_{P,t} + \lambda_t \kappa_{P,t} = g_{P,t} + \lambda \kappa_{P,t}$, (A.32) can be rewritten as

$$g_{P,t} + \lambda \kappa_{P,t} + l^{-1} - r - w_t^i \sigma_{P,t}^2 - \frac{\lambda_t \kappa_{P,t}}{1 - w_t^i \kappa_{P,t}} = 0. \tag{A.33}$$

This is a quadratic equation in w_t^i. Again, only the smaller root is valid under the leverage constraint. Therefore,

$$(w_t^i)^* = w_t = \frac{\kappa_{P,t}A_t + \sigma_{P,t}^2 - \sqrt{(\kappa_{P,t}A_t - \sigma_{P,t}^2)^2 + 4\lambda_t(\kappa_{P,t}\sigma_{P,t})^2}}{2\kappa_{P,t}\sigma_{P,t}^2}, \tag{A.34}$$

where $A_t \equiv \hat{g}_{P,t} + \lambda_t \kappa_{P,t} + l^{-1} - r = g_{P,t} + \lambda \kappa_{P,t} + l^{-1} - r$. ∎

Proof of Proposition 2. First, from the optimal consumption in (A.30), the evolution of the optimal per-capita wealth of the speculators is

$$dW_t/W_t = (r - \rho + w_t A_t)dt + w_t \sigma_{P,t}d\omega_t - w_t \kappa_{P,t}dN_t. \tag{A.35}$$

Applying Ito's lemma to both sides of $x_t = W_t/D_t$ gives

$$\begin{aligned} dx_t/x_t = &(r - \rho + w_t A_t - g_D - \lambda\kappa - w_t\sigma_D\sigma_{P,t} + \sigma_D^2)dt \\ &+ (w_t\sigma_{P,t} - \sigma_D)d\omega_t + (\kappa - w_t\kappa_{P,t})(1-\kappa)^{-1}dN_t. \end{aligned} \tag{A.36}$$

Matching terms between (A.36) with (A.27) then gives

$$g_{x,t} + \lambda \kappa_{x,t} = \hat{g}_{x,t} + \lambda_t \kappa_{x,t} = (r - \rho + w_t A_t - g_D - \lambda \kappa - w_t \sigma_D \sigma_{P,t} + \sigma_D^2) x_t,$$
$$\sigma_{x,t} = (w_t \sigma_{P,t} - \sigma_D) x_t, \qquad \kappa_{x,t} = (w_t \kappa_{P,t} - \kappa)(1 - \kappa)^{-1} x_t. \tag{A.37}$$

Next, differentiating both sides of $P_t/D_t = l(x_t, \lambda_t)$ by Ito's lemma and applying Eqs. (1), (2), (21), and (A.27) yields

$$g_{P,t} + \lambda \kappa_{P,t} - g_D - \lambda \kappa + \sigma_D^2 - \sigma_D \sigma_{P,t}$$
$$= (l_x/l)(g_{x,t} + \lambda \kappa_{x,t}) + (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)\sigma_{x,t}^2,$$
$$\sigma_{P,t} - \sigma_D = (l_x/l)\sigma_{x,t}, \tag{A.38}$$
$$(1 - \kappa)^{-1}(1 - \kappa_{P,t})l(x_t, \lambda_t) = l((1 - \kappa)^{-1}(1 - w_t \kappa_{P,t})x_t, \lambda_t + b(\lambda_t)).$$

The two arguments of function l and its derivatives, x_t and λ_t, are sometimes omitted for brevity.

Rewriting the market clearing condition in (30) as

$$w_t(x_t, \lambda_t) = (c_0 l(x_t, \lambda_t) - c_1)x_t^{-1} \tag{A.39}$$

and combining it with (A.37) and (A.38), I derive the Brownian volatility and the average growth rate of the risky asset in the absence of a crash as

$$\sigma_{P,t} = \frac{1 - (l_x/l)x_t}{1 - (l_x/l)w_t x_t}\sigma_D = \frac{1 - (l_x/l)x_t}{1 - (l_x/l)(c_0 l - c_1)}\sigma_D, \tag{A.40}$$

$$
\begin{aligned}
g_{P,t} + \lambda \kappa_{P,t} = \bar{g}_{P,t} &= [1 - (l_x/l)w_t x_t]^{-1}\big\{g_D + \lambda \kappa - \sigma_D^2 + \sigma_D \sigma_{P,t} \\
&\quad + (l_x/l)x_t[r - \rho - g_D - \lambda \kappa + \sigma_D^2 + w_t(l^{-1} - r - \sigma_D \sigma_{P,t})] \\
&\quad + (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)\sigma_{x,t}^2\big\} \\
&= [1 - (l_x/l)(c_0 l - c_1)]^{-1}\big\{g_D + \lambda \kappa - \sigma_D^2 + \sigma_D \sigma_{P,t} \\
&\quad + (l_x/l)x_t[r - \rho - g_D - \lambda \kappa + \sigma_D^2 + (l^{-1} - r - \sigma_D \sigma_{P,t})(c_0 l - c_1)/x_t] \\
&\quad + (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)[(l/l_x)(\sigma_{P,t} - \sigma_D)]^2\big\}.
\end{aligned}
\tag{A.41}
$$

Substituting (A.39) into the first-order condition of (A.33) leads to

$$\kappa_{P,t} = \frac{\bar{g}_{P,t} + l^{-1} - r - w_t^i \sigma_{P,t}^2}{\lambda_t + (\bar{g}_{P,t} + l^{-1} - r - w_t^i \sigma_{P,t}^2)w_t^i} = \frac{\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t}{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t}. \tag{A.42}$$

Combining (A.42) and the last equation in (A.38) gives

$$
\begin{aligned}
&\frac{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t]((c_0 l - c_1)/x_t - 1)}{(1 - \kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}l(x_t, \lambda_t) \\
&= l\left(\frac{\lambda_t}{(1 - \kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}x_t, \lambda_t + b(\lambda_t)\right),
\end{aligned}
\tag{A.43}
$$

where $\sigma_{P,t}$ and $\bar{g}_{P,t}$ are derived in (A.40) and (A.41). The arguments, sometimes omitted, of function l are x_t and λ_t. Eq. (A.43) is a second-order partial differential-difference equation for l.

For the boundary conditions, first note that as $x_t = W_t/D_t \to \infty$, (A.39) implies that $w_t \to 0$. The first-order condition in (A.32) then implies

$$\lim_{x_t \to \infty} l(x_t, \lambda_t) = [r - \lim_{x_t \to \infty} \hat{g}_{P,t}(x_t, \lambda_t)]^{-1} \equiv m(\lambda_t) \tag{A.44}$$

for $\forall \lambda_t \in (\lambda_m, \lambda_h)$, which reduces (A.43) to

$$m'(\lambda_t)[a(\lambda_t) - \lambda_t b(\lambda_t)] = m(\lambda_t)\left(\lambda_t - g_D - \lambda\kappa - m^{-1}(\lambda_t) + r - \frac{(1-\kappa)\lambda_t m(\lambda_t + b(\lambda_t))}{m(\lambda_t)}\right), \tag{A.45}$$

a first-order differential-difference equation. Normally, one boundary condition is required to solve this type of equation. However, in this particular case, no boundary condition should be imposed because $a(\lambda_t) - \lambda_t b(\lambda_t)$ goes to zero as λ_t goes to λ_m so λ_h is a singular point for this differential-difference equation.[27]

As $x_t = W_t/D_t \to 0$, the long-term investors are the only holders of the risky asset. From (A.34)

$$w_t(0, \lambda_t) = \frac{\kappa A + \sigma_D^2 - \sqrt{(\kappa A - \sigma_D^2)^2 + 4\lambda_t(\kappa\sigma_D)^2}}{2\kappa\sigma_D^2}. \tag{A.46}$$

By Eqs. (A.39) and (A.46)

$$\lim_{x_t \to 0} l(x_t, \lambda_t) = c_1/c_0 \tag{A.47}$$

$$\lim_{x_t \to 0} l_\lambda(x_t, \lambda_t) = w_t(0, \lambda_t)/c_0 \tag{A.48}$$

for $\forall \lambda_t \in (\lambda_m, \lambda_l)$.

Eqs. (A.39) to (A.42) and Eq. (26) verify that the evolutions of $g_{P,t}$, $\sigma_{P,t}$, $\kappa_{P,t}$, w_t, and $\hat{g}_{P,t}$ are governed by the evolutions of x_t and λ_t, which, along with (A.37), verify that x_t and λ_t are jointly Markov. ∎

3. Solving the Equilibrium for the General Model in Section IV

Proof of Lemma 3. The proof is the same as the proof of Lemma 2, except that the true crash severity $\kappa_{P,t}$ is now replaced by the perceived crash severity $\hat{\kappa}_{P,t}$. ∎

Proof of Proposition 3. Define the value function for speculator i as

$$J(W_t^i, x_t, \lambda_t) \equiv \max_{\{C_{t+s}^i, w_{t+s}^i \leq \hat{\kappa}_{P,t+s}^{-1}\}_{s \geq 0}} \mathbb{E}_t^i\left[\int_0^\infty e^{-\rho t}\ell n(C_{t+s}^i)ds\right], \tag{A.49}$$

---

[27]For a formal treatment of singular boundary value problems, see Powers (2006).

Write the perceived evolution of the state variable x_t generically as

$$dx_t \hat{=} (\hat{g}_{x,t} + \lambda_t \hat{\kappa}_{x,t})dt + \sigma_{x,t}d\omega_t - \hat{\kappa}_{x,t}dN_t \hat{=} (g_{x,t} + \lambda \kappa_{x,t})dt + \sigma_{x,t}d\omega_t - \hat{\kappa}_{x,t}dN_t, \quad \text{(A.50)}$$

where $\hat{g}_{x,t}$ is the perceived conditional expected rate of change of the state variable x_t, and $\hat{\kappa}_{x,t}$ is the perceived percentage reduction of x_t upon a crash. The true evolution of x_t, on the other hand, is

$$dx_t = (g_{x,t} + \lambda \kappa_{x,t})dt + \sigma_{x,t}d\omega_t - \kappa_{x,t}dN_t, \quad \text{(A.51)}$$

where $\kappa_{x,t}$ is the true percentage reduction of x_t upon a crash.

Given the optimal consumption in (50), the perceived evolution of the optimal per-capita wealth of the speculators is

$$dW_t/W_t \hat{=} (r - \rho + w_t A_t)dt + w_t \sigma_{P,t}d\omega_t - w_t \hat{\kappa}_{P,t}dN_t. \quad \text{(A.52)}$$

Applying Ito's lemma under speculators' beliefs to both sides of $x_t = W_t/D_t$ gives

$$dx_t/x_t \hat{=} [g_D + \lambda\kappa - r + \rho - w_t A_t + w_t^2 \sigma_{P,t}^2 - w_t \sigma_D \sigma_{P,t}]dt$$
$$+ (\sigma_D - w_t \sigma_{P,t})d\omega_t - (\kappa - w_t \hat{\kappa}_{P,t})(1 - w_t \hat{\kappa}_{P,t})^{-1}dN_t. \quad \text{(A.53)}$$

Matching terms between (A.50) and (A.53) then gives

$$g_{x,t} + \lambda \kappa_{x,t} = \hat{g}_{x,t} + \lambda_t \hat{\kappa}_{x,t} = (r - \rho + w_t A_t - g_D - \lambda\kappa - w_t \sigma_D \sigma_{P,t} + \sigma_D^2)x_t,$$
$$\sigma_{x,t} = (w_t \sigma_{P,t} - \sigma_D)x_t, \qquad \hat{\kappa}_{x,t} = (w_t \hat{\kappa}_{P,t} - \kappa)(1-\kappa)^{-1}x_t. \quad \text{(A.54)}$$

Differentiating both sides of $P_t/D_t = l(x_t, \lambda_t)$ using Ito's lemma and Eqs. (1), (2), (21), and (A.51) gives

$$g_{P,t} + \lambda\kappa_{P,t} - g_D - \lambda\kappa + \sigma_D^2 - \sigma_D\sigma_{P,t}$$
$$= (l_x/l)(g_{x,t} + \lambda\kappa_{x,t}) + (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)\sigma_{x,t}^2,$$
$$\sigma_{P,t} - \sigma_D = (l_x/l)\sigma_{x,t},$$
$$(1-\kappa)^{-1}(1 - \kappa_{P,t})l(x_t, \lambda_t) = l(x_t - \kappa_{x,t}, \lambda_t + b(\lambda_t)). \quad \text{(A.55)}$$

The two arguments of function l and its derivatives are sometimes omitted for brevity. Note that unlike in the proof for Proposition 2, the last equation in (A.54) and the last equation in (A.55) cannot be linked directly since the former contains the perceived crash size of x_t and the latter contains the true crash size. Also note that (A.55) is from the view of an outside econometricians, not from the speculators' perception.

Rewrite the market clearing condition in (49) as

$$w_t(x_t, \lambda_t) = (c_0 l(x_t, \lambda_t) - c_1)x_t^{-1}. \quad \text{(A.56)}$$

By (A.54), (A.55), and (A.39), I derive the Brownian volatility and the average growth rate of the risky asset in the absence of a crash as

$$\sigma_{P,t} = \frac{1 - (l_x/l)x_t}{1 - (l_x/l)w_t x_t}\sigma_D = \frac{1 - (l_x/l)x_t}{1 - (l_x/l)(c_o l - c_1)}\sigma_D, \tag{A.57}$$

$$
\begin{aligned}
g_{P,t} + \lambda\kappa_{P,t} = \bar{g}_{P,t} = {}&[1 - (l_x/l)w_t x_t]^{-1}\left\{g_D + \lambda\kappa - \sigma_D^2 + \sigma_D\sigma_{P,t}\right. \\
&+ (l_x/l)x_t[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + w_t(l^{-1} - r - \sigma_D\sigma_{P,t})] \\
&\left.+ (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)\sigma_{x,t}^2\right\} \\
= {}&[1 - (l_x/l)(c_0 l - c_1)]^{-1}\left\{g_D + \lambda\kappa - \sigma_D^2 + \sigma_D\sigma_{P,t}\right. \\
&+ (l_x/l)x_t[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + (l^{-1} - r - \sigma_D\sigma_{P,t})(c_0 l - c_1)/x_t] \\
&\left.+ (l_\lambda/l)[a(\lambda_t) - \lambda_t b(\lambda_t)] + \tfrac{1}{2}(l_{xx}/l)[(l/l_x)(\sigma_{P,t} - \sigma_D)]^2\right\}.
\end{aligned}
\tag{A.58}
$$

Substituting the first-order condition of (51) and the equilibrium portfolio weight of (A.56) into the assumption in (44) for boundedly rational speculators gives

$$
\begin{aligned}
&\frac{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t]((c_0 l - c_1)/x_t - 1)}{(1-\kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}l(x_t, \lambda_t) \\
&= l\left(\left[\xi + \frac{(1-\xi)\lambda_t}{(1-\kappa)\{\lambda_t + [\bar{g}_{P,t} + l^{-1} - r - \sigma_{P,t}^2(c_0 l - c_1)/x_t](c_0 l - c_1)/x_t\}}\right]x_t, \lambda_t + b(\lambda_t)\right),
\end{aligned}
\tag{A.59}
$$

where $\sigma_{P,t}$ and $\bar{g}_{P,t}$ are solved in (A.57) and (A.58). The arguments, sometimes omitted, of function l and its derivatives are x_t and λ_t. Eq. (A.59) is a second-order partial differential equation for l. It is important to note that the all the equilibrium quantities, except the actual realized crash severity, are determined purely from the perceptions of speculators and banks.

As $x_t = W_t/D_t \to \infty$, (A.56) implies that $w_t \to 0$. Then the first-order condition in (51) implies

$$\lim_{x_t\to\infty} l(x_t, \lambda_t) = [r - \lim_{x_t\to\infty}\hat{g}_{P,t}(x_t, \lambda_t)]^{-1} \equiv m(\lambda_t) \tag{A.60}$$

for $\forall\lambda_t \in (\lambda_m, \lambda_h)$. In this limiting case, (A.59) reduces to

$$m'(\lambda_t)(a(\lambda_t) - \lambda_t b(\lambda_t)) = m(\lambda_t)\left(\lambda_t - g_D - \lambda\kappa - m^{-1}(\lambda_t) + r - \frac{(1-\kappa)\lambda_t m(\lambda_t + b(\lambda_t))}{m(\lambda_t)}\right), \tag{A.61}$$

a first-order differential-difference equation. Eq. (A.45) is identical to Eq. (A.61) because bounded rationality plays little role when speculators invest most of their wealth in the safe asset and hence experience no deleveraging in the case as $x_t \to \infty$.

As $x_t \to 0$, the long-term investors are the only holders of the risky asset. From (52)

$$w_t(0, \lambda_t) = \frac{\kappa A + \sigma_D^2 - \sqrt{(\kappa A - \sigma_D^2)^2 + 4\lambda_t(\kappa\sigma_D)^2}}{2\kappa\sigma_D^2}. \tag{A.62}$$

Eqs. (A.56) and (A.62) imply

$$\lim_{x_t \to 0} l(x_t, \lambda_t) = c_1/c_0 \tag{A.63}$$

$$\lim_{x_t \to 0} l_\lambda(x_t, \lambda_t) = w_t(0, \lambda_t)/c_0 \tag{A.64}$$

for $\forall \lambda_t \in (\lambda_m, \lambda_l)$. Eqs. (A.62) to (A.64) are identical to Eqs. (A.46) to (A.48) because bounded rationality plays little role when the population fraction for speculators goes to zero.

To determine the true crash severity on the asset price $\kappa_{P,t}$, note that there are two separate cases. First, if $w_t \leq 1$, speculators lose the fraction $\kappa_{P,t}$ of their risky asset investment upon a crash, so $\kappa_{x,t} = (w_t \kappa_{P,t} - \kappa)x_t/(1 - \kappa)$. Second, if $w_t > 1$, speculators have borrowed $(w_t - 1)W_t$ from banks and need to pay back $(w_t - 1) \min[1, (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})]W_t$ upon a crash, so $\kappa_{x,t} = \{(1 - w_t) \max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})] - \kappa + w_t \kappa_{P,t}\}x_t/(1 - \kappa)$. Substituting these two values of $\kappa_{x,t}$ back into the last equation in (A54) yields

$$\frac{1 - \kappa_{P,t}}{1 - \kappa} l(x_t, \lambda_t) = \begin{cases} l\left(\dfrac{1 - w_t \kappa_{P,t}}{1 - \kappa} x_t, \lambda_t + b(\lambda_t)\right) & w_t \leq 1 \\ l\left(\dfrac{1 - w_t \kappa_{P,t} + (w_t - 1) \max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})]}{1 - \kappa} x_t, \lambda_t + b(\lambda_t)\right) & w_t > 1 \end{cases} . \tag{A.65}$$

If $w_t \leq 1$, the realized post-crash wealth level is $(1 - w_t \kappa_{P,t})W_t$, while the anticipated post-crash wealth level is $(1 - w_t \hat{\kappa}_{P,t})W_t$. If $w_t > 1$, the realized post-crash wealth level is $\{1 - w_t \kappa_{P,t} + (w_t - 1) \max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})]\}W_t$, while the anticipated post-crash wealth level is again $(1 - w_t \hat{\kappa}_{P,t})W_t$. Given this, the unanticipated percentage loss for the speculators is

$$UPL^S(x_t, \lambda_t) = \begin{cases} w_t(\kappa_{P,t} - \hat{\kappa}_{P,t}) & w_t \leq 1 \\ w_t(\kappa_{P,t} - \hat{\kappa}_{P,t}) - (w_t - 1) \max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})] & w_t > 1 \end{cases} . \tag{A.66}$$

If $w_t \leq 1$, no bank funding is involved in the risky asset investment, so $UPL^B = 0$. If $w_t > 1$, the dollar amount paid back to banks after a crash is $(w_t - 1) \min[1, (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})]W_t$, and the amount banks anticipated to get back is $(w_t - 1)W_t$. Given this, the unanticipated percentage loss for banks is

$$UPL^B(x_t, \lambda_t) = \begin{cases} 0 & w_t \leq 1 \\ \max[0, 1 - (1 - \kappa_{P,t})\theta/(1 - \hat{\kappa}_{P,t})] & w_t > 1 \end{cases} . \tag{A.67}$$

∎

B. The Numerical Procedure for Solving the Equilibrium

1. Solving the Ordinary Differential-Difference Equation for the Rational Benchmark

The second-order differential-difference equation in (15) includes a jump term; the equation not only links the values of l at adjacent points of x_t but also connects them to the function evaluated at a distant point, $(1 - \kappa)^{-1}(1 - w_t \kappa_{P,t})x_t$, with the jump size for x_t endogenously determined in equilibrium. Due to this complexity, the standard finite-difference approach is insufficient for solving the problem. Instead, I use a projection method with Chebyshev polynomials. For brevity, below I omit the subscript t in all variables.

The value of x ranges from 0 to ∞, whereas the domain for Chebyshev polynomials is $[-1, 1]$, so I transform x to a new state variable z

$$z = (x - \gamma)/(x + \gamma). \tag{B.1}$$

Define $h(z) \equiv l(x(z))$ and rewrite the differential-difference equation as

$$\frac{(1 - \kappa_P)h(z)}{1 - \kappa} = h\left(\frac{(1 + z) - \gamma^{-1}\kappa_P(c_0 h - c_1)(1 - z) - (1 - \kappa)(1 - z)}{(1 + z) - \gamma^{-1}\kappa_P(c_0 h - c_1)(1 - z) + (1 - \kappa)(1 - z)}\right), \tag{B.2}$$

where

$$\sigma_P = \frac{1 - \frac{1}{2}(h'/h)(1 - z^2)}{1 - \frac{1}{2\gamma}(h'/h)(c_0 h - c_1)(1 - z)^2}\sigma_D, \tag{B.3}$$

$$
\begin{aligned}
\bar{g}_P = &[1 - \tfrac{1}{2\gamma}(h'/h)(c_0 l - c_1)(1 - z)^2]^{-1}\left\{g_D + \lambda\kappa + \sigma_D\sigma_P - \sigma_D^2\right. \\
&+ \tfrac{1}{2}(h'/h)(1 - z^2)[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + \gamma^{-1}(h^{-1} - r - \sigma_D\sigma_P)(c_0 h - c_1)(1 - z)/(1 + z)] \\
&+ \left.\tfrac{1}{2}[h''/h - 2h'/(h(1 - z))][(\sigma_P - \sigma_D)(h/h')]^2\right\},
\end{aligned}
\tag{B.4}
$$

$$\kappa_P = \frac{\bar{g}_P + h^{-1} - r - \gamma^{-1}\sigma_P^2(c_0 h - c_1)(1 - z)/(1 + z)}{\lambda + \gamma^{-1}[\bar{g}_P + h^{-1} - r - \gamma^{-1}\sigma_P^2(c_0 h - c_1)(1 - z)/(1 + z)](c_0 h - c_1)(1 - z)/(1 + z)}. \tag{B.5}$$

Reexpress the boundary conditions in Eqs. (A.18), (A.19), and (A.21) as

$$\lim_{z \to 1} h(z) = (r - g_D)^{-1}, \tag{B.6}$$

$$\lim_{z \to -1} h(z) = (r - g_D)^{-1}[1 + k/(1 - \mu)]^{-1}, \tag{B.7}$$

$$\lim_{z \to -1} h'(z) = \gamma w(0)/(2c_0). \tag{B.8}$$

The objective is to numerically solve equation (B.2) subject to the boundary conditions in Eqs. (B.6) to (B.8). To apply the projection method, note from (B.7) and (B.8) that the unknown

function $h(z)$ can be approximated by

$$\hat{h}(z) = \frac{c_1}{c_0}\left(1 + \frac{\gamma w(0)}{2c_1}(1+z)\right) + (1+z)^2 \sum_{r=0}^n a_r T_r(z), \tag{B.9}$$

where $T_r(z)$ is the rth degree Chebyshev polynomial of the first kind.[28] The projection method chooses the coefficients $\{a_r\}_{r=0}^n$ so that the differential-difference equation and boundary conditions are *approximately* satisfied. One criterion for a good approximation is a minimum weighted sum of squared errors

$$\sum_{i=1}^M \frac{1}{\sqrt{1-z_i^2}}\left[\frac{(1-\hat{\kappa}_P)\hat{h}(z_i)}{1-\kappa} - \hat{h}\left(\frac{(1+z_i) - \gamma^{-1}\hat{\kappa}_P(c_0\hat{h}-c_1)(1-z_i) - (1-\kappa)(1-z_i)}{(1+z_i) - \gamma^{-1}\hat{\kappa}_P(c_0\hat{h}-c_1)(1-z_i) + (1-\kappa)(1-z_i)}\right)\right]^2$$
$$+ K[\hat{h}(1) - (r-g_D)^{-1}]^2, \tag{B.10}$$

where $\{z_i\}_{i=0}^M$ are the M zeros of $T_M(z)$. The last term in (B.10) guarantees that the boundary condition in (B.6) can be approximately satisfied.[29] By the Chebyshev interpolation theorem, if M is sufficiently larger than n and if the sum of weighted square in (B.10) is sufficiently small, the approximated h function in (B.9) should be very close to the true solution.

For the numerical results in Section II of the main text, I set $\gamma = 30$, $n = 25$, $M = 200$, and $K = 10^6$. I then apply the Levenberg-Marquardt algorithm and obtain a minimized sum of squared errors less than 10^{-6}. The small size of the errors indicates a good convergence of the numerical solution. The solution is also insensitive to the choice of M or K. These findings indicate that the numerical solution is a sufficient approximation for the true h function.

2. Solving the Partial Differential-Difference Equation for the General Model in Section III

In this section, I solve the second-order partial differential-difference equation in (A.43) subject to the boundary conditions in (A.44), (A.47), and (A.48).

First, $m(\lambda_t)$ in (A.44) needs to be solved numerically. I first transform the belief variable λ_t to a new variable $y_t \equiv [2\lambda_t - (\lambda_h + \lambda_m)]/(\lambda_h - \lambda_m)$ with the required range $[-1, 1]$. Now define $n(y) \equiv m(\lambda(y))$ and rewrite (A.45) as

$$n'(y)[a(y) - \lambda(y)b(y)] = \tfrac{1}{2}(\lambda_h - \lambda_m)n(y)\left(\lambda(y) - g_D - \lambda\kappa - n^{-1}(y) + r - \frac{(1-\kappa)\lambda(y)n(y')}{n(y)}\right), \tag{B.11}$$

---

[28]See Mason and Handscomb (2003) for detailed discussion of the properties of Chebyshev polynomials.

[29]Note that $T_n(1) = 1$, $T_n(-1) = (-1)^n$, $T_n'(1) = n^2$, and $T_n'(-1) = (-1)^{n-1}n^2$.

where

$$\lambda(y) = \tfrac{1}{2}[y(\lambda_h - \lambda_m) + (\lambda_h + \lambda_m)],$$
$$a(y) = \tfrac{1}{2}[(1 - y)q_l - (y + 1)q_h](\lambda_h - \lambda_m) - q_h(\lambda_m - \lambda_l),$$
$$b(y) = [y(\lambda_h - \lambda_m) + (\lambda_h + \lambda_m)]^{-1}(1 - y)(\lambda_h - \lambda_m)[\tfrac{1}{2}(y + 1)(\lambda_h - \lambda_m) + (\lambda_m - \lambda_l)],$$
$$y' = [2(\lambda(y) + b(y)) - (\lambda_h + \lambda_m)]/(\lambda_h - \lambda_m).$$

(B.12)

The singular boundary value problem in (B.11) does not require a boundary condition. I apply the same projection method as discussed in Appendix B.1 and approximate n by

$$\hat{n}(y) = \sum_{r=0}^{n} b_r T_r(y).$$

(B.13)

I then choose $\{b_r\}_{r=0}^{n}$ to minimize the weighted sum of squared errors

$$\sum_{i=1}^{M} \frac{1}{\sqrt{1 - y_i^2}} \left[\begin{array}{l} \hat{n}'(y_i)(a(y_i) - \lambda(y_i)b(y_i)) - \tfrac{1}{2}(\lambda_h - \lambda_m)\hat{n}(y_i) \\ \times \left(\lambda(y_i) - g_D - \lambda\kappa - \hat{n}^{-1}(y_i) + r - \dfrac{(1 - \kappa)\lambda(y_i)\hat{n}(y'_i)}{n(y_i)} \right) \end{array} \right]^2,$$

(B.14)

with y_i chosen as the M zeros of $T_M(y)$. I set $n = 25$ and $M = 200$, apply the Levenberg-Marquardt algorithm, and obtain a solution with a minimized sum of residual errors of (B.14) less than 10^{-14}.

To further solve the two-dimensional system, again transform x to z defined in (B.1) and write $h(z, y) \equiv l(x(z), \lambda(y))$. Rewrite the differential-difference equation of (A.43) for h as

$$\frac{(1 - \kappa_P)h(z, y)}{1 - \kappa} = h\left(\frac{(1 + z) - \gamma^{-1}\kappa_P(c_0 h - c_1)(1 - z) - (1 - \kappa)(1 - z)}{(1 + z) - \gamma^{-1}\kappa_P(c_0 h - c_1)(1 - z) + (1 - \kappa)(1 - z)}, \frac{2[\lambda(y) + b(y)] - (\lambda_m + \lambda_h)}{\lambda_h - \lambda_m} \right),$$

(B.15)

where

$$\sigma_P = \frac{1 - \tfrac{1}{2}(h_z/h)(1 - z^2)}{1 - \tfrac{1}{2\gamma}(h_z/h)(c_0 h - c_1)(1 - z)^2} \sigma_D,$$

(B.16)

$$\bar{g}_P = [1 - \tfrac{1}{2\gamma}(h_z/h)(c_0 h - c_1)(1 - z)^2]^{-1} \Big\{ g_D + \lambda\kappa + \sigma_D\sigma_P - \sigma_D^2$$
$$+ \tfrac{1}{2}(h_z/h)(1 - z^2)[r - \rho - g_D - \lambda\kappa + \sigma_D^2 + \gamma^{-1}(h^{-1} - r - \sigma_D\sigma_P)(c_0 h - c_1)(1 - z)/(1 + z)]$$
$$+ \tfrac{2}{\lambda_h - \lambda_m}(h_y/h)[a(y) - \lambda(y)b(y)] + \tfrac{1}{2}[h_{zz}/h - 2h_z/(h(1 - z))][(\sigma_P - \sigma_D)(h/h_z)]^2 \Big\},$$

(B.17)

$$\kappa_P = \frac{\bar{g}_P + h^{-1} - r - \gamma^{-1}\sigma_P^2(c_0 h - c_1)(1 - z)/(1 + z)}{\lambda(y) + \gamma^{-1}[\bar{g}_P + h^{-1} - r - \gamma^{-1}\sigma_P^2(c_0 h - c_1)(1 - z)/(1 + z)](c_0 h - c_1)(1 - z)/(1 + z)}.$$

(B.18)

The boundary conditions in Eqs. (A.44), (A.47), and (A.48) become

$$\lim_{z \to 1} h(z, y) = n(y) \tag{B.19}$$

$$\lim_{z \to -1} h(z, y) = c_1/c_0 \tag{B.20}$$

$$\lim_{z \to -1} h_z(z, y) = \gamma w(y)/(2c_0) \tag{B.21}$$

for $\forall y \in (-1, 1)$, with $w(y) = \frac{\kappa A + \sigma_D^2 - \sqrt{(\kappa A - \sigma_D^2)^2 + 4\lambda(y)(\kappa \sigma_D)^2}}{2\kappa \sigma_D^2}$.

Provided the boundary conditions in (B.20) and (B.21), the unknown function $h(z, y)$ can be approximated by

$$\hat{h}(z, y) = \frac{c_1}{c_0}\left(1 + \frac{\gamma w(y)}{2c_1}(1 + z)\right) + (1 + z)^2 \sum_{i+j \leq n} a(i, j)T_i(z)T_j(y). \tag{B.22}$$

The boundary condition in (B.19) is approximated by

$$\frac{c_1}{c_0}\left(1 + \frac{\gamma w(y)}{c_1}\right) + 4 \sum_{i+j \leq n} a(i, j)(-1)^i T_j(y) = \hat{n}(y) \tag{B.23}$$

for $\forall y \in (-1, 1)$, where $\hat{n}(y)$ is the numerical solution of (B.14).

Now choose $\{a(i, j)\}_{i+j \leq n}$ such that $\hat{h}(z, y)$ approximately satisfies (B.15) and (B.19). A good approximation is to minimize the weighted sum of squared errors

$$\sum_{i,j=1}^{M} w(i,j) \left[\frac{(1 - \hat{\kappa}_P(z_i, y_j))\hat{h}(z_i, y_j)}{1 - \kappa} - \hat{h}\left(\frac{(1 + z_i) - \gamma^{-1}\hat{\kappa}_P(z_i, y_j)(c_0\hat{h}(z_i, y_j) - c_1)(1 - z_i) - (1 - \kappa)(1 - z_i)}{(1 + z_i) - \gamma^{-1}\hat{\kappa}_P(z_i, y_j)(c_0\hat{h}(z_i, y_j) - c_1)(1 - z_i) + (1 - \kappa)(1 - z_i)}, \frac{2[\lambda(y_j) + b(y_j)] - (\lambda_m + \lambda_h)}{\lambda_h - \lambda_m} \right) \right]^2$$

$$+ K \sum_{j=1}^{M} \frac{1}{\sqrt{1 - y_j^2}}[\hat{h}(1, y_j) - \hat{n}(y_j)]^2, \tag{B.24}$$

where $w(i, j) = [(1 - z_i^2)(1 - y_j^2)]^{-1/2}$, and z_i, y_j are the M zeros of T_M. The last term in (B.24) ensures that the boundary condition in (B.19) is approximately satisfied. There are $(n+1)(n+2)/2$ unknown coefficients. By the Chebyshev interpolation theorem, if M^2 is sufficiently larger than $(n + 1)(n + 2)/2$ and if the weighted sum of squared errors in (B.24) is sufficiently small, the approximated h function in (B.22) should be close to the true solution.

For numerical results in Section III of the main text, I again choose $\gamma = 30$, and then I set $n = 40$, $M = 90$, and $K = 10^6$. In this case, there are 861 unknown coefficients and 8100 grid points. I apply the Levenberg-Marquardt algorithm and obtain a minimized sum of squared errors

for (B.24) of 0.027; this is equivalent to a percentage error about 0.007%. Such a small percentage error, together with the observation that the numerical solution is insensitive to changes in n, M and K, guarantees a good approximation of the true h function by $\hat{h}$.

3. Solving the Partial Differential-Difference Equation for the General Model in Section IV

The procedure for solving the price-dividend ratio as a function of the two state variables is by and large the same as the one outlined in Appendix B.2. Instead of (B.24), I now minimize

$$
\sum_{i,j=1}^{M} w(i,j) \left[\hat{h} \left(\begin{array}{c} \dfrac{(1 - \hat{\kappa}_P(z_i, y_j))\hat{h}(z_i, y_j)}{1 - \kappa} - \\[2mm] \dfrac{(1 - \xi\kappa)(1 + z_i) - (1 - \xi)\gamma^{-1}\hat{\kappa}_P(z_i, y_j)(c_0\hat{h}(z_i, y_j) - c_1)(1 - z_i) - (1 - \kappa)(1 - z_i)}{(1 - \xi\kappa)(1 + z_i) - (1 - \xi)\gamma^{-1}\hat{\kappa}_P(z_i, y_j)(c_0\hat{h}(z_i, y_j) - c_1)(1 - z_i) + (1 - \kappa)(1 - z_i)}, \\[2mm] \dfrac{2[\lambda(y_j) + b(y_j)] - (\lambda_m + \lambda_h)}{\lambda_h - \lambda_m} \end{array} \right) \right]^2
$$
$$
+ K \sum_{j=1}^{M} \frac{1}{\sqrt{1 - y_j^2}} [\hat{h}(1, y_j) - \hat{n}(y_j)]^2
$$

(B.25)

over $\{a(i,j)\}_{i+j \leq n}$. Note that in the case of full rationality on forecasting the crash severity, $\xi = 0$ and (B.25) reduces to (B.24).

For numerical results in Section IV of the main text, I choose $\gamma = 30$, $n = 40$, $M = 90$, and $K = 10^6$, and apply the Levenberg-Marquardt algorithm and obtain a minimized sum of squared errors of (B.25) less than 0.03 for $\xi = 0.1, 0.2, \ldots 1$, which is equivalent to a percentage error of less than 0.008% for all the values of ξ I examine. Such a small percentage error, together with the observation that the numerical solution is insensitive to changes in n, M and K, guarantees a good approximation of the true h function by $\hat{h}$.

For the true crash severity, I implement a simple interpolation method to search for the intersection between the left-hand side and the right-hand side of (61) with $\kappa_{P,t}$ as the running variable on each grid point.

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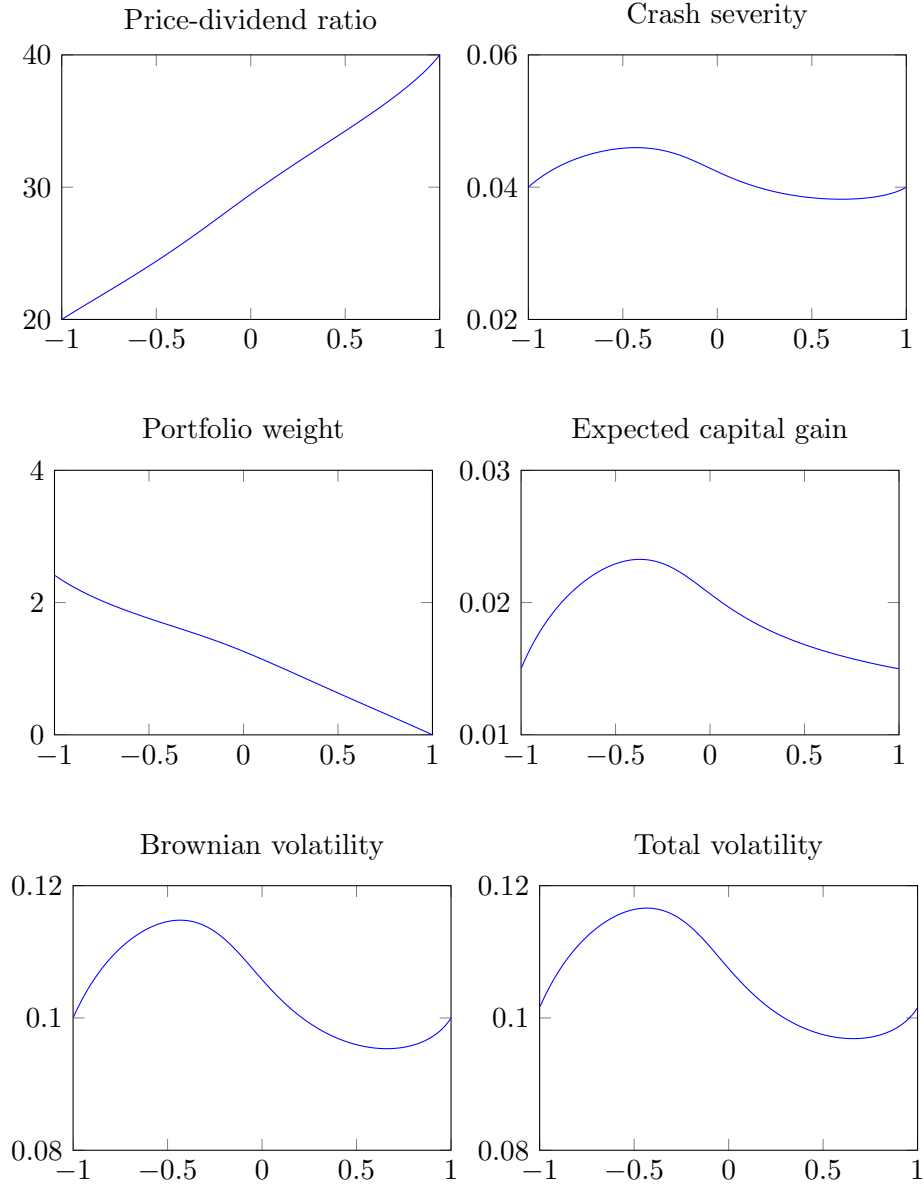
Figure 1. Asset Pricing Implications for the Rational Benchmark. This figure plots the price-dividend ratio h, the crash severity κ_P, speculators' risky asset portfolio weight w, the conditional expected capital gain of the stock g_P, the Brownian volatility σ_P, and the total volatility $\bar{\sigma}_P$, as functions of the transformed wealth-dividend ratio z defined in (19). The parameter values are: $\mu = 0.5$, $r = 4\%$, $g_D = 1.5\%$, $\rho = 1\%$, $\sigma_D = 10\%$, $k = 0.5$, $\kappa = 0.04$, and $\lambda = 0.2$. The numerical approximations follow the procedure in Appendix B.1 with $\gamma = 30$, $n = 25$, and $M = 200$.
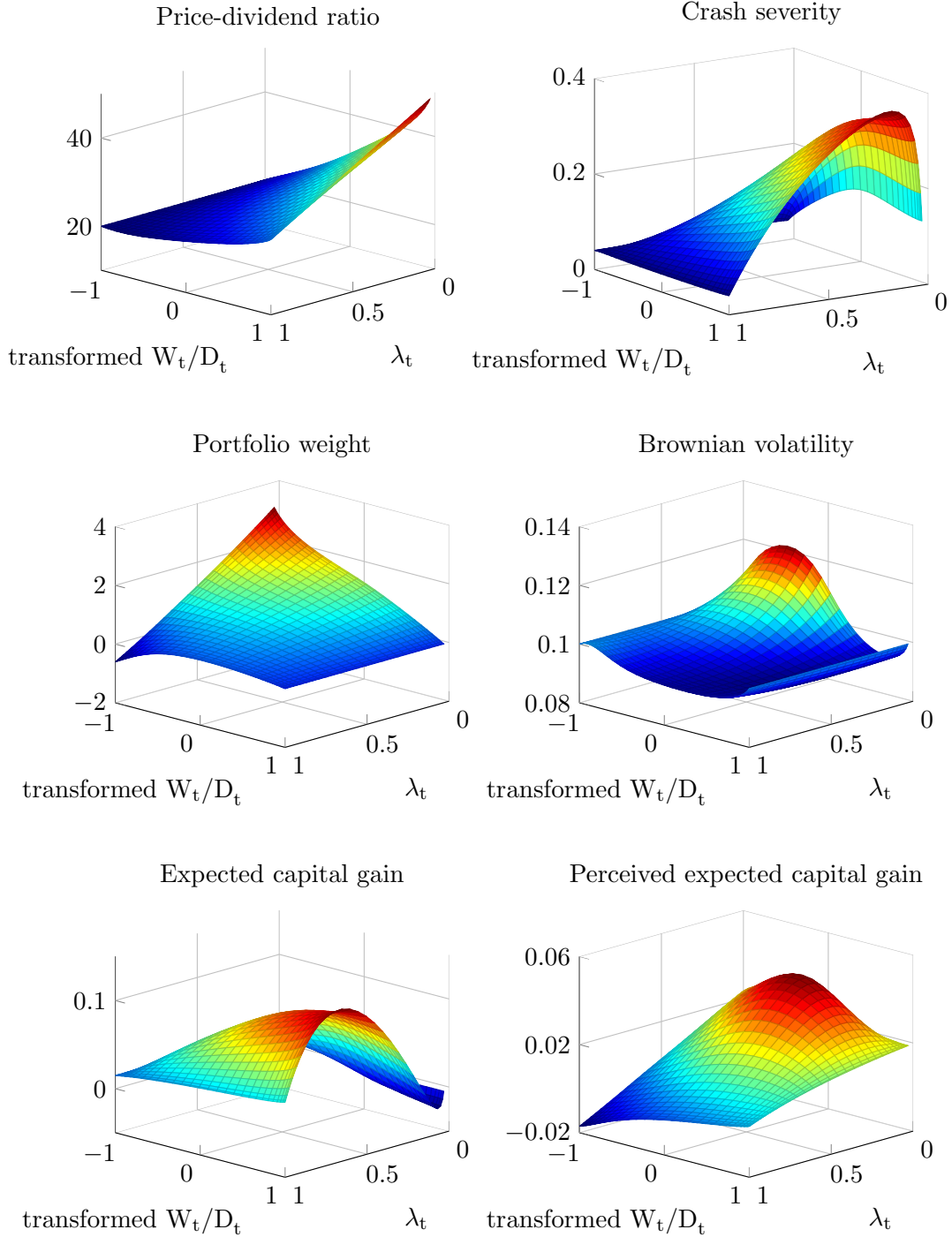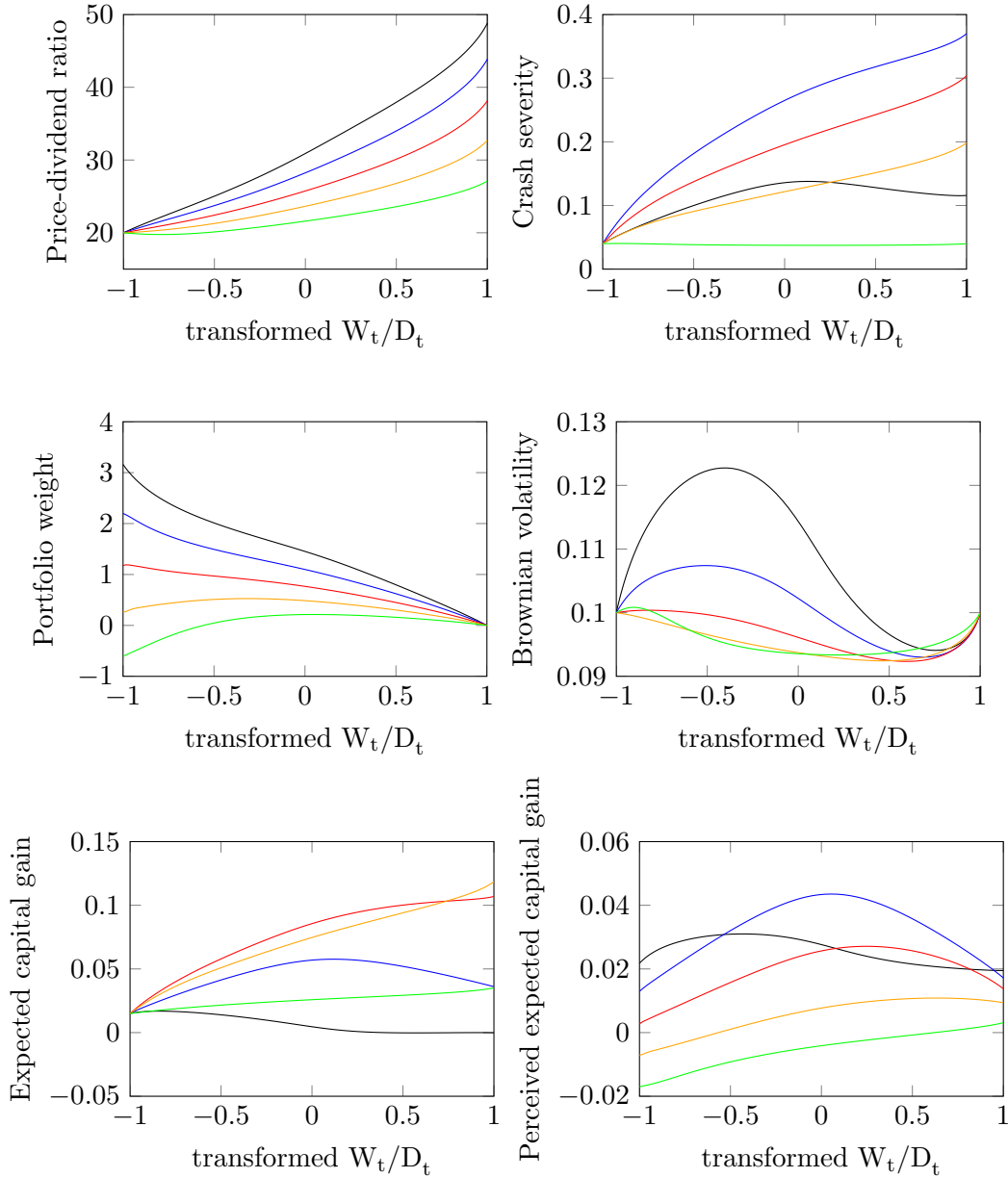
Figure 2a. Model Implications of Incorrect Beliefs about Crash Likelihood (3-D). This figure plots the price-dividend ratio h, the crash severity $\kappa_{P,t}$, the portfolio weight w_t, the Brownian volatility $\sigma_{P,t}$, the true expected capital gain $g_{P,t}$, and the perceived expected capital gain $\hat{g}_{P,t}$, as functions of λ_t and z_t. The parameter values are: $\mu = 0.5$, $r = 4\%$, $g_D = 1.5\%$, $\rho = 1\%$, $\sigma_D = 10\%$, $k = 0.5$, $\kappa = 0.04$, $\lambda = 0.2$, $\lambda_l = 0.025$, $\lambda_h = 1$, $q_h = 0.025$, and $q_l = 0.005$. The numerical approximations follow the procedure in Appendix B.2 with $\gamma = 30$, $n = 40$, and $M = 90$.

Figure 2b. Model Implications of Incorrect Beliefs about Crash Likelihood (2-D). This figure plots the price-dividend ratio h, the crash severity $\kappa_{P,t}$, the portfolio weight w_t, the Brownian volatility $\sigma_{P,t}$, the true expected capital gain $g_{P,t}$, and the perceived expected capital gain $\hat{g}_{P,t}$, as functions of z_t for various values of λ_t. The black, blue, red, orange, and green line corresponds to $\lambda_t = \lambda_m$, 0.25, 0.5, 0.75, and 1, respectively. The parameter values are the same as those for Figure 2a, and the numerical approximations follow the procedure in Appendix B.2 with $\gamma = 30$, $n = 40$, and $M = 90$.
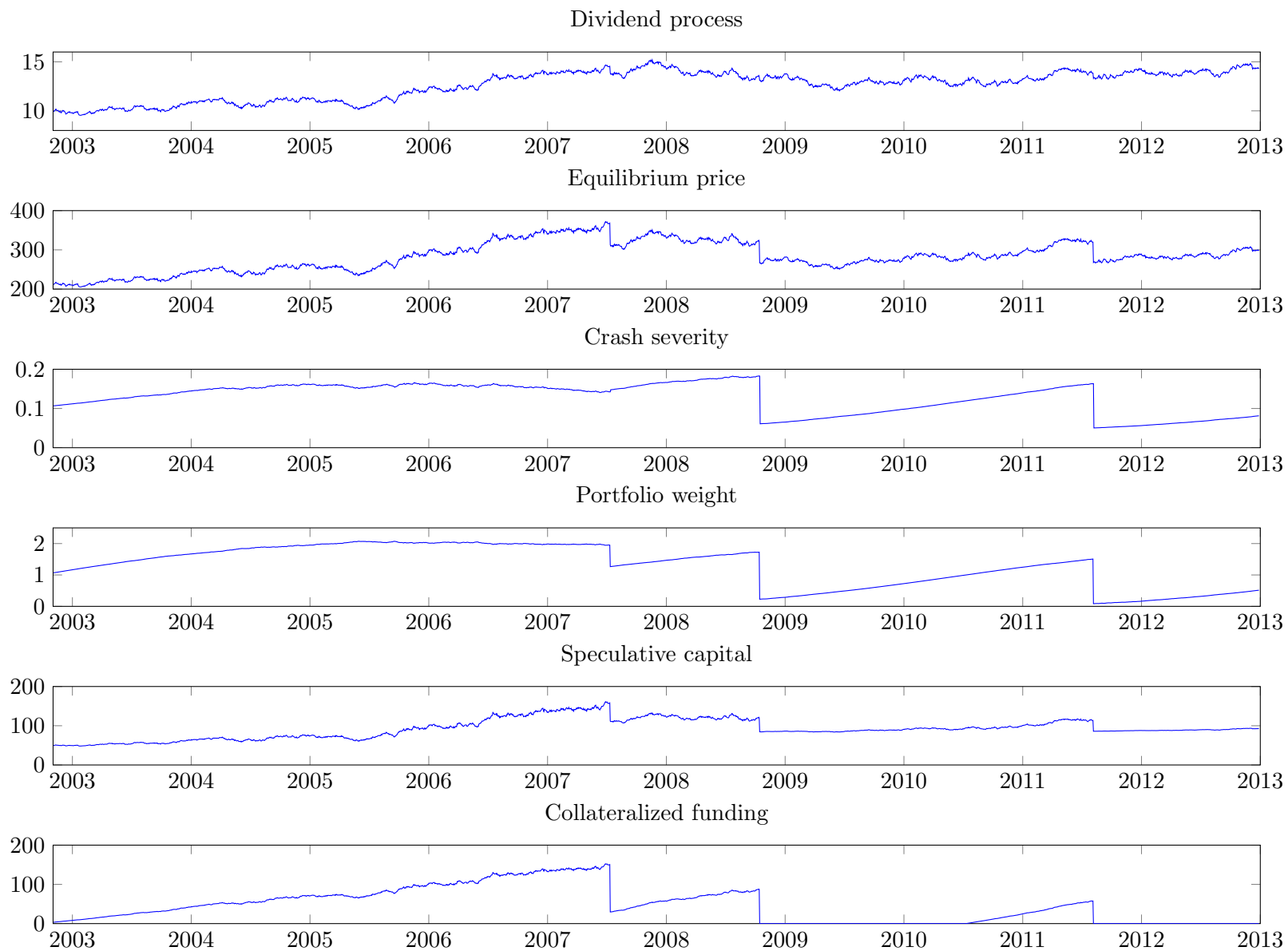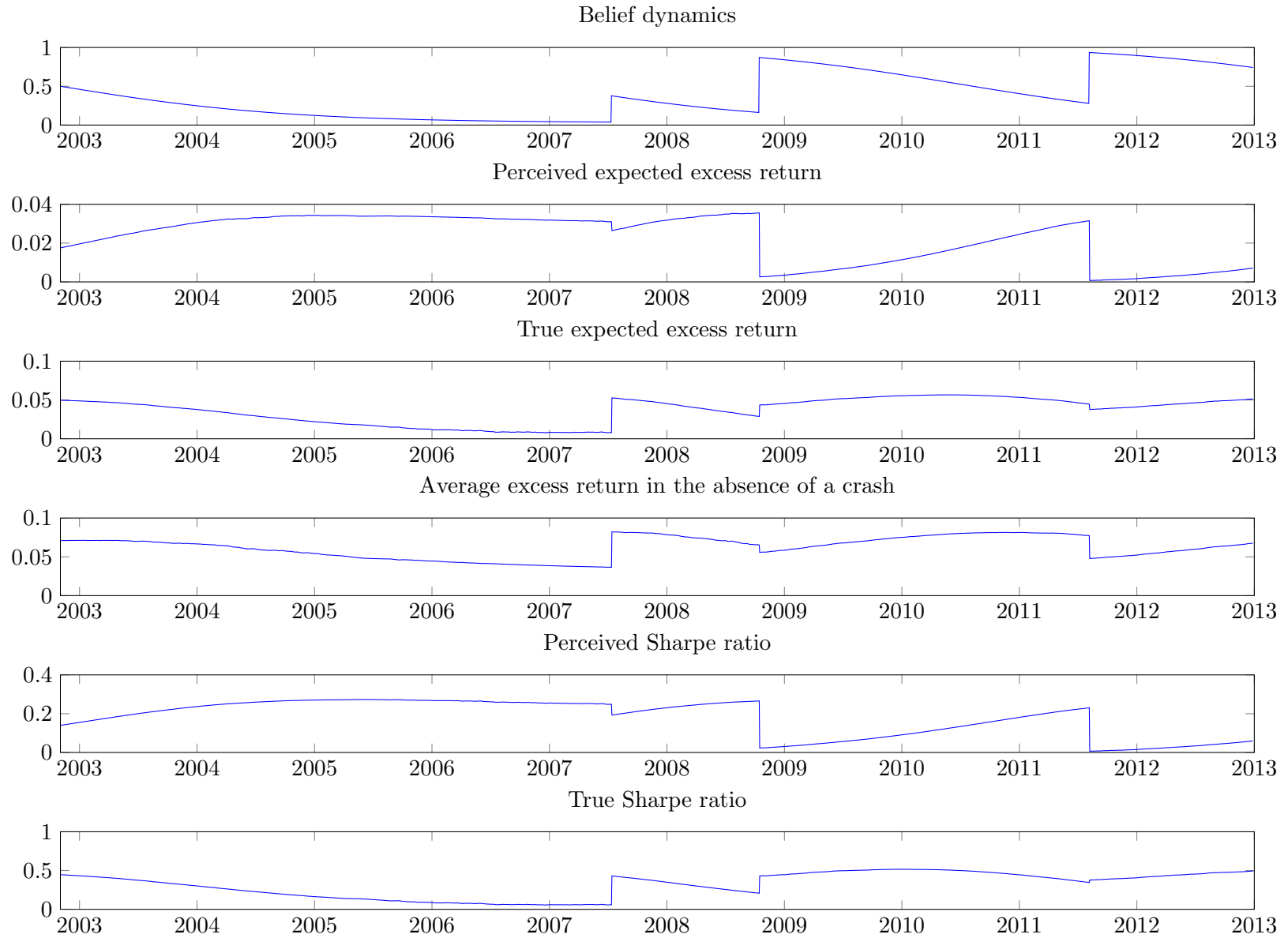
Figure 3a. Time-Series Implications based on a Simulated Dividend Process. This figure presents the time series of the equilibrium price P_t, the crash severity $\kappa_{P,t}$, the portfolio weight w_t, the per-capita wealth for speculators W_t, as well as the amount of collateralized funding $\max(w_t - 1, 0)W_t$, given a simulated dividend path D_t. The initial dividend level is 10 and the initial values for the state variables are $\lambda_t = 0.5$ and $x_t = 5$. Three Poisson shocks are set on day 1182, 1499, and 2207, respectively.

Figure 3b. Time-Series Implications based on a Simulated Dividend Process. This figure presents the time series of the belief dynamics λ_t, the perceived expected excess return $\hat{g}_{P,t} + h^{-1} - r$, the true expected excess return $\bar{g}_{P,t} - \lambda\kappa_{P,t} + h^{-1} - r$, the average excess return in the absence of a crash $\bar{g}_{P,t} + h^{-1} - r$, the perceived Sharpe ratio $[\hat{g}_{P,t} + h^{-1} - r]/(\sigma_{P,t}^2 + \lambda_t\kappa_{P,t}^2)^{1/2}$, as well as the true Sharpe ratio $[\bar{g}_{P,t} - \lambda\kappa_{P,t} + h^{-1} - r]/(\sigma_{P,t}^2 + \lambda_t\kappa_{P,t}^2)^{1/2}$, given a simulated dividend path D_t. The initial dividend level is 10 and the initial values for the state variables are $\lambda_t = 0.5$ and $x_t = 5$. Three Poisson shocks are set on day 1182, 1499, and 2207, respectively.
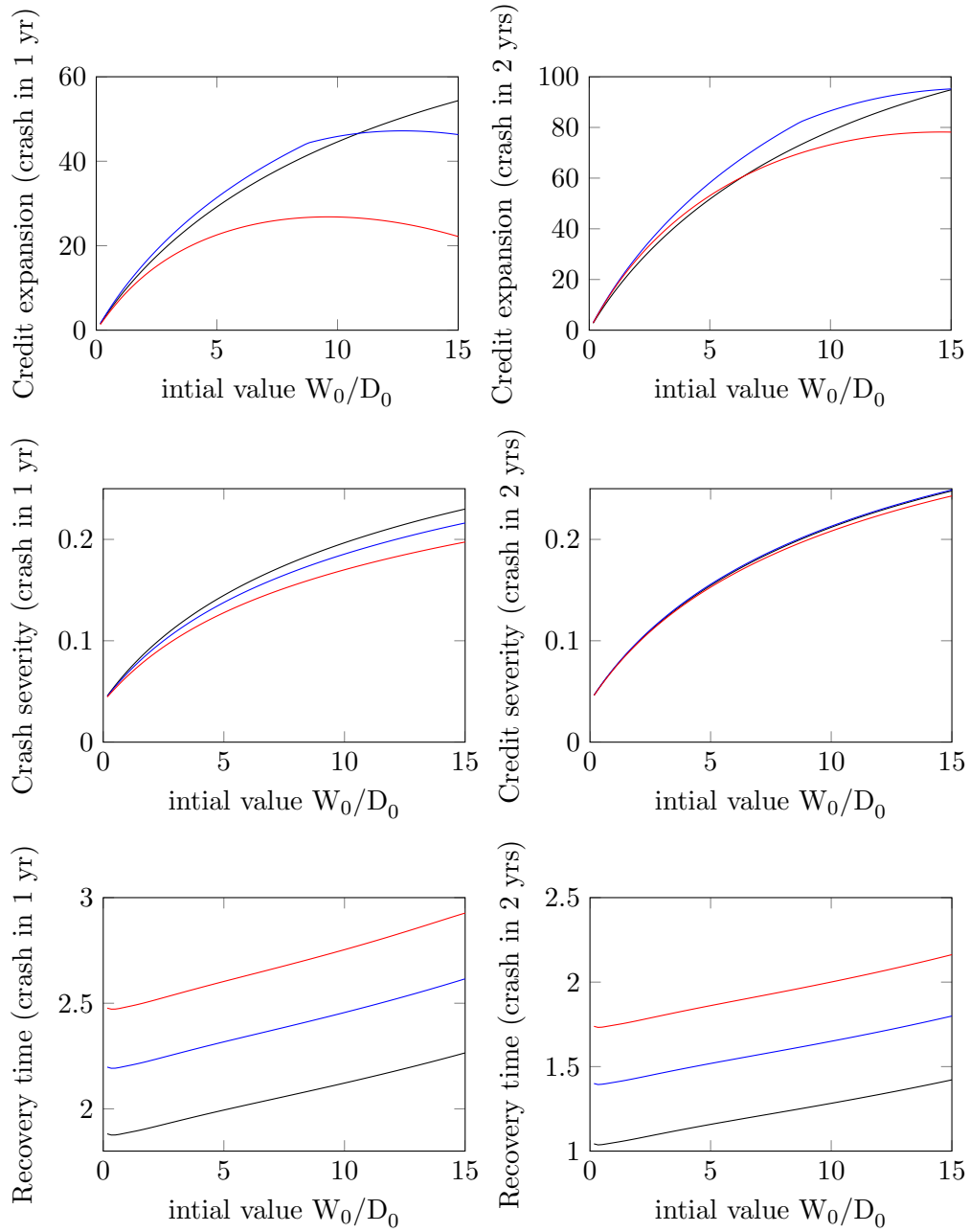
Figure 4. Time-Series Implications of Incorrect Beliefs about Crash Likelihood. This figure plots the credit expansion over a course of one year or two years in the absence of an intervening crash event, the end-of-period crash severity on the asset price, and the subsequent recovery time, with each quantity averaged over 1000 dividend paths for various initial values of W_0/D_0 and λ_0. The black, blue, and red lines correspond to $\lambda_0 = 0.4$, 0.5, and 0.6, respectively.
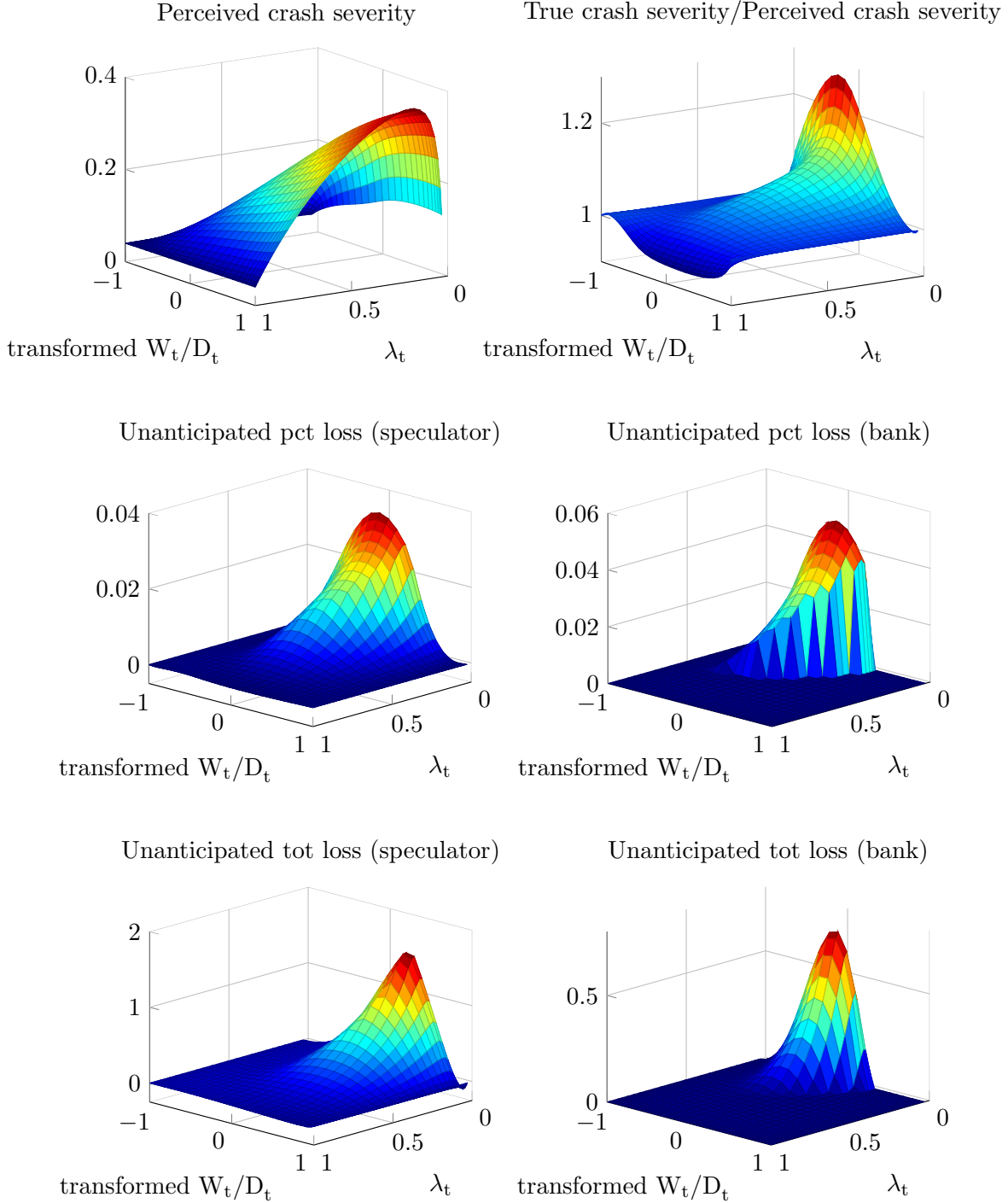
Figure 5a. Model Implications of Incorrect Beliefs about Crash Severity (3-D). This figure plots the perceived crash severity $\hat{\kappa}_{P,t}$, the ratio of the true over the perceived crash severity $\kappa_{P,t}/\hat{\kappa}_{P,t}$, the unanticipated percentage loss for speculators and for banks UPL^S and UPL^B, the unanticipated total loss for speculators and banks UTL^S and UTL^B, as functions of λ_t and z_t. The parameter values are: $\mu = 0.5$, $r = 4\%$, $g_D = 1.5\%$, $\rho = 1\%$, $\sigma_D = 10\%$, $k = 0.5$, $\kappa = 0.04$, $\lambda = 0.2$, $\lambda_l = 0.025$, $\lambda_h = 1$, $q_h = 0.025$, $q_l = 0.005$, $\xi = 1$, and $\theta = 1$. The numerical approximations follow the procedure in Appendix B.3 with $\gamma = 30$, $n = 40$, and $M = 90$.
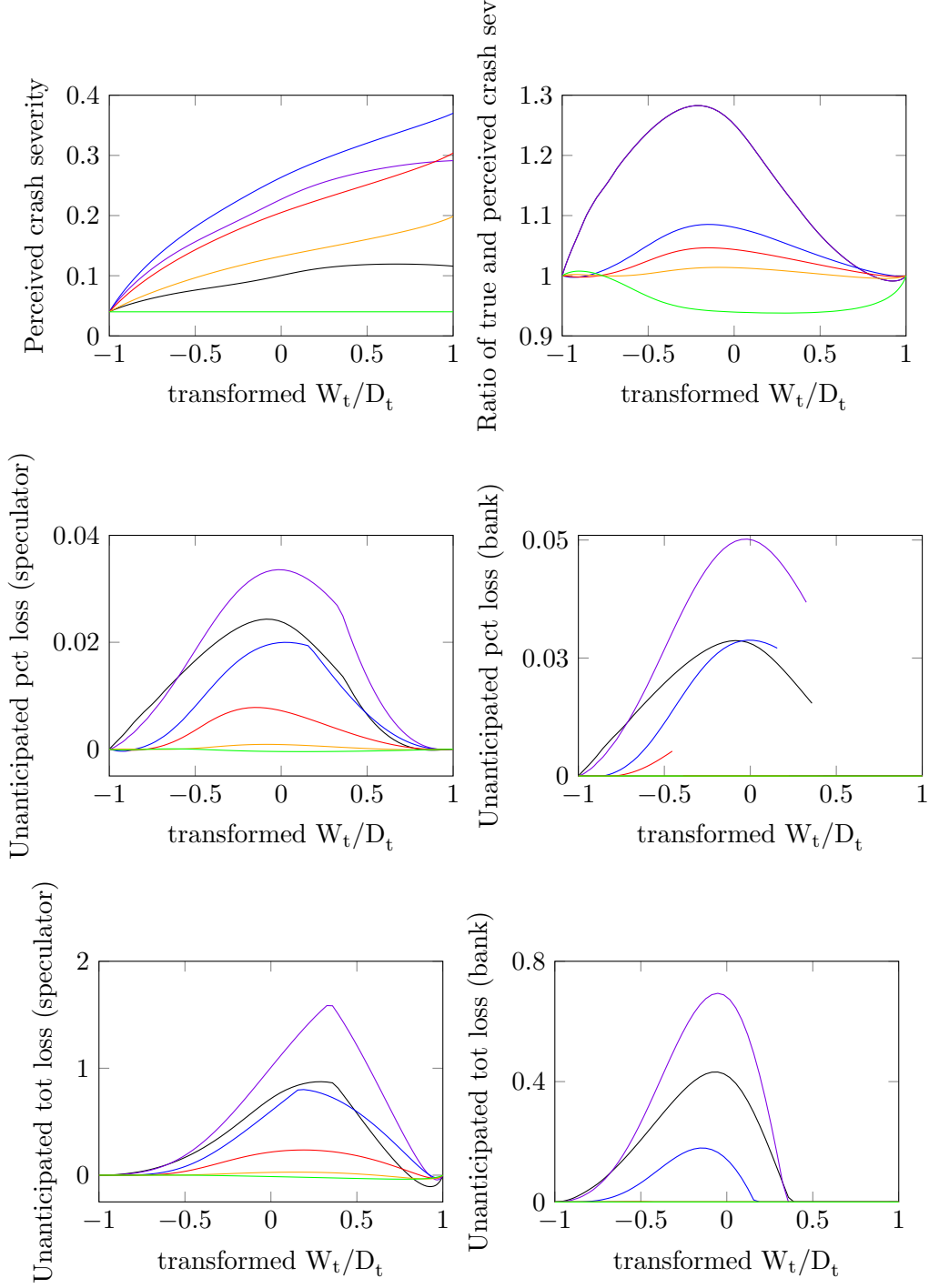
Figure 5b. Model Implications of Incorrect Beliefs about Crash Severity (2-D). This figure plots the perceived crash severity $\hat{\kappa}_{P,t}$, the ratio of the true over the perceived crash severity $\kappa_{P,t}/\hat{\kappa}_{P,t}$, the unanticipated percentage loss for speculators and for banks UPL^S and UPL^B, the unanticipated total loss for speculators and banks UTL^S and UTL^B, as functions of z_t for various values of λ_t. The black, purple, blue, red, orange, and green line corresponds to $\lambda_t = \lambda_m$, 0.0625, 0.25, 0.5, 0.75, and 1, respectively. The parameter values are the same as for Figure 5a, and the numerical approximations follow the procedure in Appendix B.3 with $\gamma = 30$, $n = 40$, and $M = 90$.
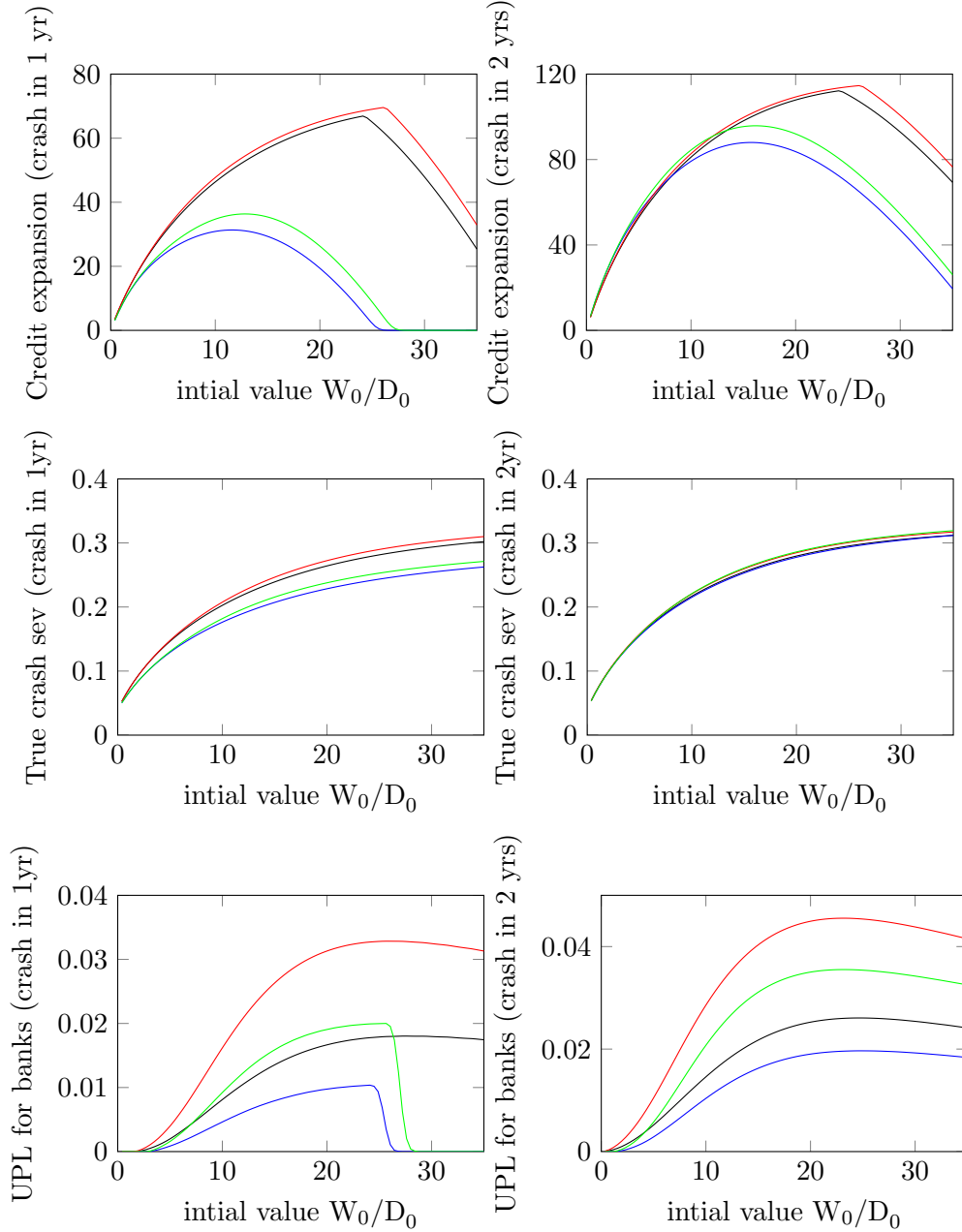
Figure 6. Time-Series Implications of Incorrect Beliefs about Crash Severity. This figure plots the credit expansion over a course of one year or two years in the absence of an intervening crash event, the end-of-period crash severity on the asset price, and the end-or-period unanticipated percentage loss for banks upon a crash, with each quantity averaged over 1000 dividend paths for various initial values of W_0/D_0 and λ_0. The black, blue, red, and green line corresponds to $(\lambda_0, \xi) = (0.4, 0.5), (0.6, 0.5), (0.4, 1)$, and $(0.6, 1)$, respectively.