A Note on the $M^B/G/1$ Queue

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Abstract

In this paper, we present a new proof of the Pollaczek-Khintchine formula for the steady state mean waiting time and the steady state moment generating function of the workload process of the $M^B/G/1$ queue with batch arrivals. Our method of proof exploits the functional Kolmogorov forward equations for the workload process of the $M^B/G/1$ queue. We also illustrate that our method of proof is much simpler and does not rely on using residual service time techniques. Finally, we also prove a batch scaling limit theorem for the $M^B/M/1$ queue. As a result, we show that the scaled queue length process converges to a workload process where the scaled batch distribution in the queue length process becomes the jump distribution in the workload process, thus illuminating a new connection between queue length processes and workload processes in the single server queue.

Keywords: Single-Server Queues, Steady State, Moments, Moment Generating Function, Functional Forward Equations, Batch Processes, Batch Scaling, Workload Processes.

1 Introduction

The M/G/1 queue is one of the most studied systems in queueing theory. It is a fundamental queueing model that has been used in a plethora of different applications including transportation, storage processes, insurance, etc. The stationary version of the $M/G/1$ workload process is an example of a Levy process with no negative jumps. This is also known as a spectrally positive Levy process. Spectrally positive Levy process have many properties and typically it is quite easy to compute their Laplace transform in closed form to generate information about their moments. These types of processes have often been studied in the insurance literature, where their a positive drift for payments and compound Poisson process representing the claims.

Some successes in studying the workload process include the paper by Abate and Whitt [1] where they study the transient behavior of the workload process and time dependent moments of the workload process. Although they mention time varying arrival rates briefly, they do not explore the effect of time varying arrival rates in much detail. In work by Choudhury et al. [6, 7], they develop an algorithm for computing the cumulative distribution function of the time dependent workload of a piecewise stationary $M_t/G_t/1$ queue. In their papers Choudhury et al. [6, 7] , they use double transform methods and they make no effort to compute the time dependent moments of the workload process in this case or make any effort to extend their approach to other types of arrival rate functions. The book Kalashnikov [15] provides a technique called the method of supplementary variables to study the transient M/G/1 queue. However, most of the results in the book are for the steady state behavior. Finally, different versions of the M/G/1 queue are studied in Boucherie and Boxma [3], Boxma et al. [5], Perry et al. [19], Boxma et al. [4], Bekker and Boxma [2], Jain and Sigman [14]

Since much is known about the M/G/1 workload process, our goal in this work is not to derive a new result about the M/G/1 workload process, but rather to give a different proof of old results. Our approach to analyzing the $M/G/1$ workload process is to exploit the functional forward equations as pioneered in Grier et al. [13], Massey and Pender [16], Pender [17, 18], which to our knowledge has not been done in the literature. We will show that we can derive a recursion for the steady state moments of the workload process and we can derive the value of the steady state moment generating function. We highlight that our method is remarkably simple and does not use any information about residual service times. Furthermore, we show how the $M/G/1$ workload process can be derived via a batch scaling limit of a $M^B/M/1$ queue, which is a new result to our knowledge.

2 The M^B/G/1 Queueing Model

In this section, we give an overview of the M^B/G/1 queueing model and introduce some of the notation that will be used throughout the rest of the paper. The M^B/G/1 queue we are considering has a single server, an unlimited waiting space, and will be operated in a first in first out (FIFO) fashion. We also assume that the arrival process is homogeneous Poisson process with arrival rate λ. The batch random variables are i.i.d with pmf $b(k)$ where $b(k) = P(B_i = k)$. The service times of the M^B/G/1 queue are i.i.d and are independent of the arrival process. Moreover, we let $\{S_{i,n} : i \geq 1, n \geq 1\}$ represent the n^{th} service time of the i^{th} batch. We assume that the service times are i.i.d and they have a continuous pdf g and cdf G and such that

$$E[S_{i,n}] \equiv \int_0^\infty y g(y) dy = \int_0^\infty \overline{G}(y) dy = \frac{1}{\mu}, \quad \text{for all } i \geq 1 \text{ and } n \geq 1. \tag{2.1}$$

Finally, when needed, we assume that the moment generating function (mgf) of the service times exists i.e. $E[e^{\theta S_{i,n}}] < \infty$ for all $i \geq 1$ and $n \geq 1$ and the mgf of the batch distribution also exists i.e. $E[e^{\theta B_i}] < \infty$.

2.1 Functional Forward Equations for the Workload Process

Although there are fluid and diffusion limit theorems for understanding the sample path behavior of the workload process, however, they are difficult to use for understanding the moment behavior of the workload process, especially the variance and higher moments. Thus, we take a different approach which relies on the functional forward equations for the workload process. Although the functional forward equations are a conditional expectation with respect to the starting value, we assume that we start at zero and omit the notation that indicates the conditional expectation. Moreover, we remove the time dependence for convenience as well. The *functional version* of the Kolmogorov forward equations for the $M^B/G/1$ workload process have the following form

$$\overset{\bullet}{E}\left[f(W(t))\right] \;=\; \lambda \cdot E\left[f(W(t)+\mathcal{S}) - f(W(t))\right] - E\left[\left(f^{'}(W(t))\right) \cdot \{W(t) > 0\}\right],$$

for all appropriate functions f and where $\mathcal{S} = \sum_{j=1}^{B} S_j$. For special cases of f such as the mean, variance, third cumulant moment, and fourth cumulant moment, we obtain the following set of cumulant moment equations:

$$\begin{aligned}
\overset{\bullet}{E}[W(t)] &= \lambda \cdot E[\mathcal{S}] - \mu \cdot E[\{W(t) > 0\}] \\
\overset{\bullet}{\mathrm{Var}}[W(t)] &= \lambda \cdot E[\mathcal{S}^2] - 2 \cdot \mathrm{Cov}\left[W(t), \{W(t) > 0\}\right] \\
\overset{\bullet}{C}^{(3)}[W(t)] &= \lambda \cdot E[\mathcal{S}^3] - 3 \cdot \mathrm{Cov}\left[\overline{W}^2(t), \{W(t) > 0\}\right] \\
\overset{\bullet}{C}^{(4)}[W(t)] &= \lambda \cdot E[\mathcal{S}^4] - 4 \cdot \left(\mathrm{Cov}\left[\overline{W}^3(t), \{W(t) > 0\}\right] - 3 \cdot \mathrm{Var}[W(t)] \cdot \mathrm{Cov}\left[W(t), \{W(t) > 0\}\right]\right)
\end{aligned}$$

where $\overline{W} \equiv W - E[W]$. For notational convenience, we will drop the dependence of the workload and queue length processes on time.

Theorem 2.1. *If we assume that $\lambda E[\mathcal{S}] < 1$, then for all $n \geq 1$ we have the following recursion for the n^{th} steady state moment of the $M^B/G/1$ workload process*

$$E[W^n] = \frac{\sum_{j=2}^{n+1} \binom{n+1}{j} E[\mathcal{S}^j] \cdot E[W^{n+1-j}]}{(n+1) \cdot (1 - \lambda \cdot E[\mathcal{S}])}. \tag{2.2}$$

Proof. We start the proof by deriving the result for the first and second moment first to give the general idea and show how to replicate the Pollaczek-Khintchine formula for the steady mean waiting time. Then, we extend the result to an arbitrary integer moment. For the first moment, we have that

$$\overset{\bullet}{E}[W] \;=\; \lambda \cdot E[\mathcal{S}] - E[\{W > 0\}].$$

Thus, in steady state, we observe that

$$\begin{aligned}
0 &= \lambda \cdot E[\mathcal{S}] - E[\{W > 0\}] \\
&= \lambda \cdot E[\mathcal{S}] - \mathbb{P}(W > 0).
\end{aligned}$$

This implies that

$$\begin{aligned}
0 &= \lambda \cdot E[\mathcal{S}] - E[\{W > 0\}] \\
\mathbb{P}(W > 0) &= \lambda \cdot E[\mathcal{S}] = \rho.
\end{aligned}$$

Now for the second moment, we have that

$$\overset{\bullet}{E}[W^2] \;=\; \lambda \cdot E[\mathcal{S}^2] + 2\lambda E[W] \cdot E[\mathcal{S}] - \mathbb{E}[W \cdot \{W > 0\}).$$

Now looking at the steady state, we have that

$$\begin{aligned} 0 &= \lambda \cdot E[\mathcal{S}^2] + 2\lambda E[W] \cdot E[\mathcal{S}] - \mathbb{E}[W \cdot \{W > 0\}]). \\ &= \lambda \cdot E[\mathcal{S}^2] + 2\lambda E[W] \cdot E[\mathcal{S}] - \mathbb{E}[W] + \mathbb{E}[W \cdot \{W = 0\}]. \end{aligned}$$

Rearranging so that $E[W]$ is on one side and making the observation that $\mathbb{E}[W \cdot \{W = 0\}] = 0$, then we obtain

$$\begin{aligned} E[W] &= \frac{\lambda \cdot E[\mathcal{S}^2]}{2 - 2\lambda \cdot E[\mathcal{S}]} \\ &= \frac{\lambda \cdot E[\mathcal{S}^2]}{2(1-\rho)}. \end{aligned}$$

Not only can we obtain this result from the second moment equation, but we can also obtain it from the variance equation

$$\overset{\bullet}{\mathrm{Var}}[W] \;=\; \lambda \cdot E[\mathcal{S}^2] - 2 \cdot \mathrm{Cov}\left[W, \{W > 0\}\right].$$

By setting the left hand side to zero, we obtain

$$\lambda \cdot E[\mathcal{S}^2] \;=\; 2 \cdot \mathrm{Cov}\left[W, \{W > 0\}\right],$$

thus it remains to solve for $\mathrm{Cov}\left[W, \{W > 0\}\right]$.

$$\begin{aligned} \mathrm{Cov}\left[W, \{W > 0\}\right] &= E[W \cdot \{W > 0\}] - E[W] \cdot \mathbb{P}(W > 0) \\ &= E[W] \cdot \mathbb{P}(W = 0) \\ &= E[W] \cdot \mathbb{P}(W = 0). \end{aligned}$$

Rearranging yields the Pollaczek-Khintchine formula again.

Now that we have seen the idea of the proof for calculating the first moment, we can extend our idea to higher moments using the same technique. For the n^{th} moment we have that

$$\overset{\bullet}{E}[W^n] \;=\; \lambda \cdot E[(W + \mathcal{S})^n - W^n] - nE[W^{n-1} \cdot \{W > 0\}].$$

Since W and $\mathcal{S}$ are independent, we have that

$$\begin{aligned} \overset{\bullet}{E}[W^n] &= \lambda \cdot E\left[\sum_{j=0}^{n} W^{n-j} \cdot \mathcal{S}^j - W^n\right] - nE[W^{n-1}] + nE[W^{n-1} \cdot \{W = 0\}] \\ &= \lambda \cdot \sum_{j=0}^{n} E\left[W^{n-j} \cdot \mathcal{S}^j - W^n\right] - nE[W^{n-1}] + nE[W^{n-1} \cdot \{W = 0\}] \\ &= \lambda \cdot \sum_{j=0}^{n} E\left[W^{n-j}\right] \cdot E\left[\mathcal{S}^j\right] - E\left[W^n\right] - nE[W^{n-1}] + nE[W^{n-1} \cdot \{W = 0\}] \\ &= \lambda \cdot \sum_{j=1}^{n} E\left[W^{n-j}\right] \cdot E\left[\mathcal{S}^j\right] - nE[W^{n-1}] + nE[W^{n-1} \cdot \{W = 0\}]. \end{aligned}$$

Now in steady state we have that

$$0 \;=\; \lambda \cdot \sum_{j=1}^{n} E\left[W^{n-j}\right] \cdot E\left[\mathcal{S}^j\right] - nE[W^{n-1}] + nE[W^{n-1} \cdot \{W = 0\}].$$

For all $n \geq 2$, we have that $E[W^{n-1} \cdot \{W = 0\}] = 0$. Thus, by isolating $E[W]$ on one side we have

$$E[W^{n-1}] = \frac{\sum_{j=2}^{n} \binom{n}{j} E[\mathcal{S}^j] \cdot E[W^{n-j}]}{n(1 - \lambda \cdot E[\mathcal{S}])}, \tag{2.3}$$

which proves our result. □

Theorem 2.2. *The steady state moment generating function for the $M^B/G/1$ workload process has the following expression*

$$E[e^{\theta \cdot W}] \;=\; \frac{\theta \cdot (1 - \rho)}{\theta - \lambda \cdot (E[e^{\theta \cdot \mathcal{S}}] - 1)}.$$

Proof.

$$\begin{aligned}\dot{E}[e^{\theta \cdot W}] &= \lambda \cdot E[e^{\theta \cdot \mathcal{S}} - 1] \cdot E[e^{\theta \cdot W}] - \theta \cdot E[e^{\theta \cdot W} \cdot \{W > 0\}] \\ &= \lambda \cdot E[e^{\theta \cdot \mathcal{S}} - 1] \cdot E[e^{\theta \cdot W}] - \theta \cdot E[e^{\theta \cdot W}] + \theta \cdot E[e^{\theta \cdot W} \cdot \{W = 0\}].\end{aligned}$$

Now by isolating $E[e^{\theta \cdot W}]$ on one side in steady state and observing that $E[e^{\theta \cdot W} \cdot \{W = 0\}] = E[\{W = 0\}] = \mathbb{P}(W = 0) = 1 - \rho$, we obtain

$$E[e^{\theta \cdot W}] \;=\; \frac{\theta \cdot (1 - \rho)}{\theta - \lambda \cdot (E[e^{\theta \cdot \mathcal{S}}] - 1)}.$$

□

2.2 Functional Forward Equations for M^B/M/1 Queue

In addition to the workload process, we can derive similar results for the queue length process. Next, we prove a similar recursion for the moments of the M^B/M/1 queue lenth process in steady state.

Theorem 2.3. *If we assume that $\lambda E[\mathcal{S}] < 1$, then for all $n \geq 1$ we have the following recursion for the n^{th} steady state moment of the M^B/M/1 queue*

$$E[Q^n] = \frac{\lambda \cdot E\left[\sum_{j=2}^{n+1} \binom{n+1}{j} Q^{n+1-j}\right] + \mu \cdot E\left[\sum_{j=2}^{n} \binom{n+1}{j} Q^{n+1-j} \cdot (-1)^j\right] + (-1)^{n+1} \cdot \mu(1 - \rho)}{\mu - \lambda}. \tag{2.4}$$

Proof. For the n^{th} moment we have that

$$\begin{aligned}
\overset{\bullet}{E}[Q^n] &= \lambda \cdot E[(Q+B)^n - Q^n] + \mu E[((Q-1)^n - Q^n)\cdot\{Q>0\}] \\
\overset{\bullet}{E}[Q^n] &= \lambda \cdot E\left[\sum_{j=0}^{n}\binom{n}{j}Q^{n-j}B^j - Q^n\right] + \mu \cdot E\left[\sum_{j=0}^{n}\binom{n}{j}Q^{n-j}\cdot(-1)^j - Q^n\right] \\
&+ \mu \cdot E\left[\left(\sum_{j=0}^{n}\binom{n}{j}Q^{n-j}\cdot(-1)^j - Q^n\right)\cdot\{Q=0\}\right] \\
&= \lambda \cdot \sum_{j=1}^{n}\binom{n}{j}\left(E\left[Q^{n-j}\right]\cdot E\left[B^j\right]\right) + \mu \cdot E\left[\sum_{j=1}^{n}\binom{n}{j}Q^{n-j}\cdot(-1)^j\right] \\
&- \mu \cdot E\left[\left(\sum_{j=1}^{n}\binom{n}{j}Q^{n-j}\cdot(-1)^j\right)\cdot\{Q=0\}\right] \\
&= (\lambda E\left[B\right] - \mu)E[Q^{n-1}] + \lambda \cdot \sum_{j=2}^{n}\binom{n}{j}\left(E\left[Q^{n-j}\right]\cdot E\left[B^j\right]\right) \\
&+ \mu \cdot E\left[\sum_{j=2}^{n}\binom{n}{j}Q^{n-j}\cdot(-1)^j\right] + (-1)^n\cdot\mu(1-\rho)
\end{aligned}$$

For all $n \geq 2$, we have that $E[Q^{n-1}\cdot\{Q=0\}] = 0$. Thus, by isolating $E[Q^{n-1}]$ on one side we have

$$E[Q^{n-1}] = \frac{\lambda \cdot \sum_{j=1}^{n}\binom{n}{j}\left(E\left[Q^{n-j}\right]\cdot E\left[B^j\right]\right) + \mu \cdot E\left[\sum_{j=2}^{n-1}\binom{n}{j}Q^{n-j}\cdot(-1)^j\right] + (-1)^n\cdot\mu(1-\rho)}{\mu - \lambda E\left[B\right]}, \tag{2.5}$$

which proves our result. □

As in the workload case, we also derive an expression for the moment generating function in the M^B/M/1 queue case as well.

Theorem 2.4. *The steady state moment generating function for the M^B/M/1 queue length process has the following expression*

$$E[e^{\theta \cdot Q}] = \frac{\mu\left(e^{-\theta}-1\right)\cdot(\rho-1)}{\lambda\cdot\left(E[e^{B\theta}]-1\right) + \mu\left(e^{-\theta}-1\right)}.$$

Proof.

$$\begin{aligned}
\overset{\bullet}{E}[e^{\theta\cdot Q}] &= \lambda\cdot\left(E[e^{B\theta}]-1\right)\cdot E[e^{\theta\cdot Q}] + \mu\left(e^{-\theta}-1\right)\cdot E[e^{\theta\cdot Q}\cdot\{Q>0\}] \\
&= \lambda\cdot\left(E[e^{B\theta}]-1\right)\cdot E[e^{\theta\cdot Q}] + \mu\left(e^{-\theta}-1\right)\cdot E[e^{\theta\cdot Q}] + \mu\left(e^{-\theta}-1\right)\cdot E[e^{\theta\cdot Q}\cdot\{Q=0\}] \\
&= \lambda\cdot\left(E[e^{B\theta}]-1\right)\cdot E[e^{\theta\cdot Q}] + \mu\left(e^{-\theta}-1\right)\cdot E[e^{\theta\cdot Q}] + \mu\left(e^{-\theta}-1\right)\cdot(1-\rho)
\end{aligned}$$

Now by isolating $E[e^{\theta \cdot Q}]$ on one side in steady state and observing that $E[e^{\theta \cdot Q} \cdot \{Q = 0\}] = E[\{Q = 0\}] = \mathbb{P}(Q = 0) = 1 - \rho$, we obtain

$$E[e^{\theta \cdot Q}] = \frac{\mu \left(e^{-\theta} - 1\right) \cdot (\rho - 1)}{\lambda \cdot (E[e^{B\theta}] - 1) + \mu \left(e^{-\theta} - 1\right)}.$$

□

3 Batch Scaling Limits for the $M^B/M/1$ Queue

In addition to showing new proofs for the moments and moment generating function for the queue length and workload processes of the $M/G/1$ queue, we can also show how the $M^B/M/1$ queue and the $M/G/1$ workload process are connected via a batch scaling limit. There has been recent interest in showing batch scaling results for infinite server queues in [8, 10, 9, 11, 12]. Our result below for the single server queue compliments these recent batch scaling results for infinite server queues and point processes.

Theorem 3.1. *Let X be a non-negative random variable and let $\lim_{n\to\infty} \frac{B^{(n)}}{n} \Rightarrow X$. If we define $Q_t^{(n)}$ be the n^{th} batch scaled queueing process, then we have that*

$$\frac{Q_t^{(n)}}{n} \Rightarrow \tilde{Q}_t \tag{3.6}$$

pointwise for each value of t where $\tilde{Q}_t$ is a workload process with jump sizes that have distribution according to X. Moreover, the steady state moment generating function of $\tilde{Q}$ is given by

$$\frac{\mu\theta(1-\rho)}{\mu\theta - \lambda\left(\mathbb{E}\left[e^{\theta X}\right] - 1\right)}. \tag{3.7}$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\left[e^{\frac{\theta}{n}Q_t^{(n)}}\right] &\equiv \frac{d}{dt}M^n(t,\theta) \\
&= \lambda\left(\mathbb{E}\left[e^{\frac{\theta}{n}B^{(n)}}\right] - 1\right)M^n(t,\theta) + \mu n\left(e^{\frac{-\theta}{n}} - 1\right)\mathbb{E}\left[e^{\frac{\theta}{n}Q_t^{(n)}} \cdot \left\{Q_t^{(n)} > 0\right\}\right] \\
&= \lambda\left(\mathbb{E}\left[e^{\frac{\theta}{n}B^{(n)}}\right] - 1\right)M^n(t,\theta) + \mu n\left(e^{\frac{-\theta}{n}} - 1\right)\mathbb{E}\left[e^{\frac{\theta}{n}Q_t^{(n)}} \cdot \left(1 - \left\{Q_t^{(n)} = 0\right\}\right)\right] \\
&= \lambda\left(\mathbb{E}\left[e^{\frac{\theta}{n}B^{(n)}}\right] - 1\right)M^n(t,\theta) + \mu n\left(e^{\frac{-\theta}{n}} - 1\right)\mathbb{E}\left[e^{\frac{\theta}{n}Q_t^{(n)}} \cdot \left(1 - \left\{Q_t^{(n)} = 0\right\}\right)\right] \\
&- \mu n\left(e^{\frac{-\theta}{n}} - 1\right)\mathbb{E}\left[e^{\frac{\theta}{n}Q_t^{(n)}} \cdot \left\{Q_t^{(n)} = 0\right\}\right] \\
&= \lambda\left(\mathbb{E}\left[e^{\frac{\theta}{n}B^{(n)}}\right] - 1\right)M^n(t,\theta) + \mu n\left(e^{\frac{-\theta}{n}} - 1\right)M^n(t,\theta) \\
&- \mu n\left(e^{\frac{-\theta}{n}} - 1\right)\mathbb{E}\left[\left\{Q_t^{(n)} = 0\right\}\right] \\
&= \lambda\left(\mathbb{E}\left[e^{\frac{\theta}{n}B^{(n)}}\right] - 1\right)M^n(t,\theta) + \mu n\left(e^{\frac{-\theta}{n}} - 1\right)M^n(t,\theta) \\
&- \mu n\left(e^{\frac{-\theta}{n}} - 1\right)\mathbb{P}\left(Q_t^{(n)} = 0\right) \\
&\overset{n\to\infty}{=} \lambda\left(\mathbb{E}\left[e^{\theta X}\right] - 1\right)M^\infty(t,\theta) - \mu\theta M^\infty(t,\theta) + \mu\theta\mathbb{P}\left(Q_t^{(\infty)} = 0\right).
\end{aligned}$$

Note that the right hand side is equivalent to the moment generating function of a workload process whose jumps are of size X and decays down toward the origin at rate μ. To derive the steady state result, we now set the right hand side to zero and observe that

$$M^{\infty}(\infty,\theta) = \frac{\mu\theta\mathbb{P}\left(Q_{\infty}^{(\infty)}=0\right)}{\mu\theta-\lambda\left(\mathbb{E}\left[e^{\theta X}\right]-1\right)} = \frac{\mu\theta(1-\rho)}{\mu\theta-\lambda\left(\mathbb{E}\left[e^{\theta X}\right]-1\right)}$$

and this completes the proof. □

4 Conclusion

We have shown how to derive the moments and moment generating function for the M^B/G/1 workload process and the M^B/M/1 queue. We use the functional forward equations and show that our proofs are quite simple and use no information about residual service times. We believe that the functional forward equations are a powerful tool that should be exploited more often. We also show a connection between the queue length process and a workload process by showing a batch scaling result for the single server queue.

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