Group Symmetries and Bike Sharing for $M/M/1/k$ Queueing Transience*

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Abstract

Fundamental stochastic models for studying the dynamics of bike sharing systems can be found within the transient behavior of the $M/M/1/k$ queue and related stopped processes. We develop new techniques involving group symmetries and complex analysis to obtain exact solutions for their transition probabilities. These methods are based on the underlying Markovian structure of these random processes and do not involve any indirect analysis from using generating functions or Laplace transforms. Our techniques exploit the intrinsic group symmetries for both the state spaces and the Markov generators for all these queueing systems related to the $M/M/1/k$ queue. Our results complement and extend the previous $M/M/1/k$ transient solutions given by Takács [44].

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Contents

1 Introduction

One of the simplest, fundamental, stochastic queueing models, that describes the random dynamics of a single server with a limited waiting customer capacity, is the $M/M/1/k$ queue. These features are common to communication systems as well as services that include communication, healthcare, and retail. Examples of telecommunication applications for these queues can be found in Altman and Jean-Marie [1], Pender [35], Hautphenne and Haviv [19], Upton and Tripathi [48], Gouweleeuw [17], and Righter [38]. Moreover, $M/M/1/k$ queueing analytical models of the *distributed coordination function* (DCF), have had numerous applications in the performance analysis of IEEE 802.11 networks Kosek-Szott [22], including approximating the loss probabilities for wireless networks Altman and Jean-Marie [1]. Finally, the simplicity of the $M/M/1/k$ queue makes it useful in a classroom setting Mathers [31] as a classical model for studying the queueing behavior of barbershops.

The steady state behavior of the $M/M/1/k$ queue is simply a truncated geometric distribution. Moreover, its transient behavior is not as well know but it can easily be expressed as a sum of exponential and trigonometric functions. Lajos Takács [44] was one of the first to reveal this queueing transient behavior in his pioneering book. His approach was to use techniques from linear algebra to solve for the transition probabilities. His analysis exploits finite dimensional tri-diagonal matrices and their eigenvalues. An important consequence of this work is that we can also derive the transition probabilities for the $M/M/1/\infty$ queue by taking the finite capacity queueing limit of $k \to \infty$. These limiting results reveal a connection to sums of modified Bessel functions. This limiting model also corresponds to having an unlimited number of parking docks for bikes along with impatient customers. They do not wait for future bike rental opportunities when all the docks are currently empty.

Other transient performance characteristics of the $M/M/1/k$ queue studied include Cohen [10] for the busy periods and the maximum number of customers simultaneously present in the queue during this same period for an $M/G/1/k$ queue. Moreover, Cohen [9] computes the bivariate transform of the number of customers served and number of blocked customers due to capacity constraints during a busy period. Here, they use complex analysis to represent the joint transform as a fraction of two contour integrals that involve the Laplace–Stieltjes transform of the customers' service time. This work was generalized by Rosenlund [39] by computing the trivariate transform of the busy period length, the number of customers served and the number of blocked customers during a busy period.

Currently, the $M/M/1/k$ queue has emerged as an important model for transportation systems such as bike sharing networks, see for example Faghih-Imani et al. [13], Hampshire and Marla [18], Nair et al. [32], O'Mahony and Shmoys [33], Pan et al. [34], Pender et al. [36], Tao [45], Tao and Pender [46], Vogel et al. [49], Chemla et al. [6], Raviv and Kolka [37], Li et al. [25, 26], Samet et al. [40], Li et al. [27], Calafiore et al. [5], Li et al. [28], Legros [23], Biondi et al. [4], Chung et al. [8], Huang et al. [20], Li and Fan [24]. One of the most important works in this area is Schuijbroek et al. [42], where they analyze the $M_t/M/1/k$ as a canonical model for bike sharing systems by using the differential equations given by the $M_t/M/1/k$ Kolmogorov forward equations. In particular they are interested in the probability the queue is *empty*, where no bikes are available, or the system is *full*, where no bike returns are possible. These probabilities provide insight on how often one must rebalance the system. We can then attempt to maximize the time period where customers have bikes available to pick up and docks available for dropping off a bike. Other recent work in the context of bike sharing is by Fricker et al. [16], Fricker and Gast [15], Tao and Pender [47], where they analyze a network of $M/M/1/k$ through the lens of empirical process theory. Instead of studying the network at a station level, Fricker et al. [16], Tao and Pender [47] both propose using an empirical process perspective to look at how many stations have exactly k bikes. This reduces the complexity from looking at the number of stations, which is roughly 800 in NYC to looking at the maximum dock capacity, which is roughly 50.

We are inspired by these bike sharing applications to study the transient behavior of the $M/M/1/k$ queue. The goal of this paper is to give readers a thorough and complete treatment of this stochastic model. Moreover, this transient analysis can be used for the optimization of such systems as seen in [7, 41, 50]. If viewed through the lens of bike sharing applications, our work provides many new types of queueing performance measures and explicit expressions for them. Our work complements that of Takács in that we primarily use complex analysis and place much less emphasis on linear algebraic techniques. Our strategy marries complex contour integration with standard probabilistic tools such as martingales, stopping times, and the Kolmogorov forward equations. This type of analysis grows organically out of the sample path behavior of the underlying Markovian queueing process. Using these methods, we avoid the use of classical generating functions and Laplace transforms. In the next subsection, we describe the main contributions of our work and how the rest of the paper is organized.

1.1 Main Contributions of the Paper

We make the following contributions in this work:

- We derive the transient behavior of the $M/M/1/k$ queue by applying new complex analytic and group symmetry methods directly to the underlying Markov process.
- Our derivation methods are a direct analysis of the transition probabilities that avoid the use of generating functions and Laplace transforms.
- Bike sharing stations provide a context for studying $M/M/1/k$ models of balanced systems. These are the transient states where the system station is neither empty nor full.

1.2 Organization of Paper

The rest of the paper is organized as follows. In Section 2, we show how the classical $M/M/1/k$ queue models an autonomous bike sharing station. We also discuss how modern systems, where managers intervene to rebalance the number of bikes in the station, correspond to stopped versions of this queueing process. Moreover, to solve for the $M/M/1/k$ transition probabilities, we first solve for a simpler queueing model where rental customers patient wait to instantly take a bike that is returned later. We also assume an unlimited parking docks so there is room for every returned bike. In Section 3, we call the resulting queueing model the *free process*.

Assuming a finite number of bike docks corresponds to confining this free process to the state space for the $M/M/1/k$ queue. In general, bike station managers may intervene when the system reaches one of the extreme states of an empty or full bike station. In Section 4, we model the dynamics of the system before we reach one of these states by studying the associated *absorbing process*. This corresponds to constructing a random stopping time for the free process. We can then compute the transition probabilities for this stopped process by applying state space symmetry methods to the free process transition probabilities.

Finally, in Section 5 we apply Markov generator symmetry methods to solve for the $M/M/1/k$ transition probabilities as a reflecting version of the free process. This is the stochastic model for an autonomous bike station. Finally, given all the derivations of the formulas, we summarize them in Section 6 and make our final comments on all these results in Section 7.

2 Modeling Autonomous Bike Sharing Stations

We begin by restricting ourselves to discussing the features of *autonomous* bike stations. These are the relevant dynamic actions that are purely customer driven where the manager does not intervene. We can then construct a simple queueing model that summarizes the dynamics of a bike service station.

1. One group of arriving customers *rent* bikes that are *currently parked* in one of the station docks.

2. Another group of arriving customers *return* their bikes *if* they can park them into some empty station dock.

3. There are only a *fixed* number of bike station docks.

4. All customers instantly *leave* the bike station. This includes all customers who successfully rent *or* return a bike.

2.1 Markov Model for the Number of Occupied Docks

Let $\{ Q_k(t) \mid t \geq 0 \}$ be an $M/M/1/k$ queueing process. We define it here as the stochastic evolution of the population for this specific Markovian *birth-death process*:

1. The number of bike parking docks k corresponds to a fixed threshold or upper bound on the integer population size.

2. The *bike return rate* corresponds to a constant birth rate λ, whenever the integer population size is strictly less than k. Otherwise, the rate is zero.

3. The *bike rental rate* corresponds to a constant death rate μ, whenever the integer population size is non-zero. Otherwise, the rate is zero.

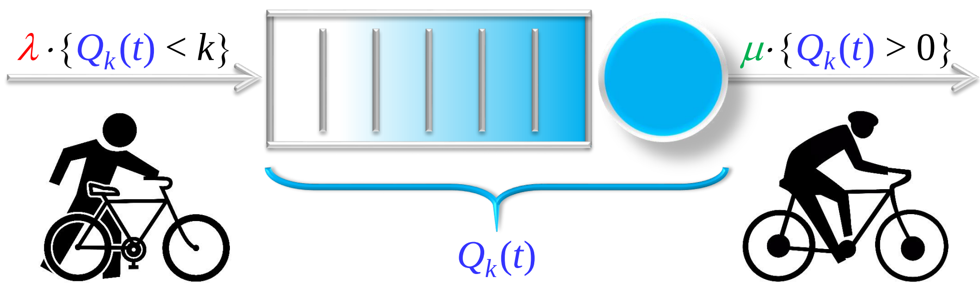

Figure 1: The $M/M/1/k$ Queue as a Bike Sharing Station Model.

The process $\{\, k - Q_k(t) \,|\, t \geq 0 \,\}$ is also an $M/M/1/k$ queueing process, where the roles of λ and μ are reversed. This models the number of *empty docks* in the same bike station.

Algebraically, this new queueing process is a special case of a *spectral conjugate* to the original Markov process. This terminology refers to a Markov generator matrix that is *similar* to some *sub-Markov* generator matrix (where there may be visible or *concealed* absorbing states) or its *transpose*. This is an equivalence relation where all the sub-Markov generator conjugates for a given generator have the same set of eigenvalues. Special cases of this spectral symmetry include:

1. The time reversal of a Markov process.

2. The stochastic dual of a Markov process on an ordered state space.

3. The "particle - anti-particle" duality for a finite capacity queue.

We summarize our discussions by the following diagram:

Bike Station	Notation	$M/M/1/k$ Queue
Initial Bike Number	$Q_k(0)$	Initial Queueing State
Bike Return Rate	λ	Arrival Rate
Bike Rental Rate	μ	Service Rate
Dock Number	k	System Capacity
Docked Bike Numbers	$\{\, Q_k(t) \,\|\, t \geq 0 \,\}$	Queueing Process
Empty Dock Numbers	$\{\, k - Q_k(t) \,\|\, t \geq 0 \,\}$	Conjugate Queueing Process
Rebalance Time	$T = \min\{\, t \,\|\, Q_k(t) = 0 \text{ or } k \,\}$	First Time to State 0 or k

Table 1: Notation in Bike Sharing and Queueing Perspectives.

2.2 Stopping Time Model for the Time to Station Rebalance

A Markovian model has a simple descriptor for the dynamics of what happens next. We call this the *state* of the system. For our model, the state is the current number of bikes parked in one of the k docks. Customers arriving to the station are successful when they either choose to return a bike to an empty dock or rent a some bike parked in occupied dock. All such bike station states are said to be *balanced*.

When the station is in an *empty* state, then a bike rental is not possible. When it is in a *full* state, then another bike return is not possible. For a non-autonomous bike station, these are times when a station manager might intervene and add or subtract bikes to *rebalance* the station.

We can model the *time to rebalance* as a *random stopping time* defined by the sample paths of our $M/M/1/k$ queueing process. This is the first time that the queueing process visits state 0 or state k. We can then determine the probability distribution for the time to rebalance by finding the exact solution for the transient distribution to the stopped version of the queueing process.

Deriving an a exact formula for the transient distribution of this time restricted process is actually simpler than deriving the transition probabilities for $M/M/1/k$ queueing process. In fact, we do the latter by showing that it has a spectral conjugate that is a variant of this stopped process.

3 Free Process Analysis

Suppose that our bike sharing station has an unlimited number of parking docks. This would make all customers returning a bike successful. If all the arrival rental customers were infinitely patient, they would just wait outside the station until a new bike is returned.

Starting with a finite number of bikes waiting for customers always yields a finite number of bikes available for rentals or a finite number of customers waiting for bikes to rent. We call the corresponding birth-death model a *free process*. This gives us a sense of the customer demand for both bikes and bike docks.

Viewing a waiting customer at the station as a "negative bike" our state space is the entire set of integers $\mathbb{Z}$. Moreover, this transforms the probabilistic dynamics of the free process into the difference of two independent Poisson processes

$$Z(t) \equiv Z(0) + \Pi_1(\lambda t) - \Pi_2(\mu t), \tag{1}$$

where we let $\{\, \Pi_1(\lambda t) | \, t \geq 0\}$ be a rate λ Poisson process that models the bike-return counting process and $\{\, \Pi_2(\mu t) | \, t \geq 0\}$ is an independent, rate μ Poisson process that models the for bike-rental counting process, and $Z(0)$ some integer constant.

This process is also called a "randomized random walk" on the integers according to Feller [14]. Moreover, when the free process is initialized at zero, the probability distribution of the free process is also called a Skellam distribution named after statistician John Gordon Skellam who first derived its formula Skellam [43].

We provide a Markovian state-transition diagram for the free process in Figure 2. Every integer value state for this process has two *competing exponential clocks*. This means that every state has two clocks set where their times to alarm are independent and exponentially distributed. Their two distinct rates are λ and μ. If the alarm with rate λ goes off first, then the process at some state n makes a transition to state $n+1$ that is labeled by λ. Otherwise, a transition from state n to state $n-1$ occurs with rate label μ.

Our $M/M/1/k$ queueing model is the *double barrier reflected version* of this free process, where the lower barrier state is 0 and the upper barrier state is k. The time to rebalance is the first time the queueing process visits either barrier. Moreover, this random stopping time is stochastically equivalent to the first time that the *free* process hits either the lower or upper barrier. A more realistic, non-autonomous bike sharing system rebalances the stations when they are empty or full of bikes. This makes the resulting stopped queueing process with absorbing states a more natural model for bike sharing stations.

3.1 Free Process Forward and Backward Equations

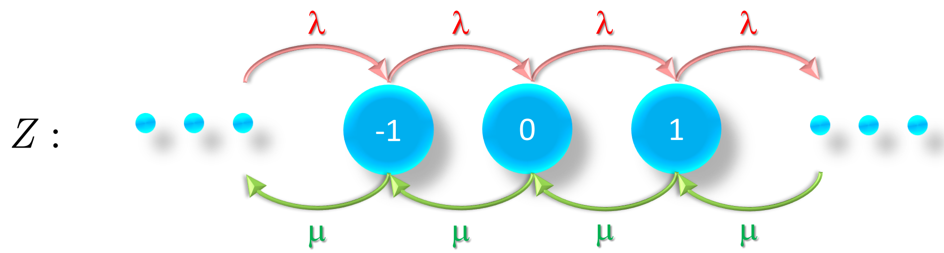

Figure 2: State Transition Diagram for the Free Process.

By differentiating the probabilities for Poisson process with respect to t, we have

$$\frac{d}{dt}\mathrm{P}\left\{\Pi(\lambda t)=n\right\}=\lambda\cdot\left(\mathrm{P}\left\{\Pi(\lambda t)=n-1\right\}-\mathrm{P}\left\{\Pi(\lambda t)=n\right\}\right) \quad (2)$$

and call them the *Kolmogorov forward equations* for the Poisson process.

We have a similar set of equations for the transition probabilities of the free process. For all initial states m and terminal states n, we have

$$\frac{d}{dt}\mathrm{P}_m\{Z(t)=n\}=\lambda\cdot\mathrm{P}_m\{Z(t)=n-1\}+\mu\cdot\mathrm{P}_m\{Z(t)=n+1\}-(\lambda+\mu)\cdot\mathrm{P}_m\{Z(t)=n\} \quad (3)$$

for all $t>0$.

These equations are equivalent to *functional versions* of the forward equations which are of the form

$$\frac{d}{dt}\mathrm{E}_m\left[f\left(Z(t)\right)\right]=\lambda\cdot\mathrm{E}_m\left[f\left(Z(t)+1\right)\right]+\mu\cdot\mathrm{E}_m\left[f\left(Z(t)-1\right)\right]-(\lambda+\mu)\cdot\mathrm{E}_m\left[f\left(Z(t)\right)\right], \quad (4)$$

for all polynomial functions f on the integers. Transforming the Kolmogorov forward equations into the functional version can be obtained by summing over the probabilities to obtain the expectations. Assuming the functional version gives us the forward equations for the transition probabilities when we set $f(Z(t))$ equal to the *indicator function* $\{Z(t)=n\}$ which is

$$\{Z(t)=n\}\equiv\begin{cases}1 & \text{when } Z(t)=n,\\ 0 & \text{when } Z(t)\neq n.\end{cases} \quad (5)$$

Hence $\{Z(t) = n\}$ is like a conditional programming statement that has the value one when the statement $Z(t) = n$ is true and zero when $Z(t) = n$ is false.

To give an explicit formula for the free process transition probabilities, we start by defining the following constants α, β, and γ:

$$\alpha = \frac{\lambda + \mu}{2}, \;\; \beta = \sqrt{\frac{\lambda}{\mu}}, \;\; \gamma = \sqrt{\lambda\,\mu}, \tag{6}$$

where α and γ, respectively, are the *arithmetic* and *geometric* means of λ and μ. Moreover, we define the complex function δ, where[1]

$$\delta(w) = \frac{1}{2} \cdot \left(w + \frac{1}{w} \right), \tag{7}$$

for all $w \in \mathbb{C}_* \equiv \mathbb{C} \setminus \{0\}$. Moreover, δ satisfies the identities

$$\delta(e^{i\theta}) = \cos\theta, \quad \delta(w) = \delta\left(\frac{1}{w}\right), \quad \text{and} \quad \alpha = \gamma \cdot \delta(\beta). \tag{8}$$

Since the arithmetic mean always exceeds its corresponding geometric mean, we always have $\delta(\beta) \geq 1$ for all real values $\beta > 0$ since $\delta(1) = 1$.

3.2 Free Transience Using Contour Integration

Theorem 3.1. *Using the constant β and the symmetric complex function δ gives us*

$$\mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(t)}\right] = \mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(0)}\right] \cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w))t}. \tag{9}$$

For all real values $r > 0$, we now have

$$\mathrm{P}_m\left\{Z(t) = n\right\} = \frac{\beta^{n-m}}{2\pi i} \cdot \oint_{|w|=r} w^{m-n} \cdot e^{-2\gamma t\cdot(\delta(\beta)-\delta(w))}\,\frac{dw}{w}. \tag{10}$$

Proof. This follows from using the functional Kolmogorov forward equations combined with the simple polynomial

$$f(n) = \left(\frac{w}{\beta}\right)^n, \tag{11}$$

where $w \in \mathbb{C}_*$ and so

$$\begin{aligned}
\frac{d}{dt}\mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(t)}\right] &= \lambda\cdot\mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(t)+1}\right] + \mu\cdot\mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(t)-1}\right] - (\lambda+\mu)\cdot\mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(t)}\right] \\
&= \left(\lambda\cdot\frac{w}{\beta} + \mu\cdot\frac{\beta}{w} - (\lambda+\mu)\right)\cdot\mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(t)}\right] \\
&= -2\gamma\cdot(\delta(\beta)-\delta(w))\cdot\mathrm{E}\left[\left(\frac{w}{\beta}\right)^{Z(t)}\right].
\end{aligned}$$

[1] The generalization of this function in Baccelli et al. [3] uses ϵ instead of δ.

The unique solution to this differential equation is Equation (9).

Using complex contour integration, the identity

$$\frac{1}{2\pi i}\oint_{|w|=r} w^n \cdot \frac{dw}{w} = \begin{cases} 1 & \text{when } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

for all real $r > 0$, gives us

$$\{Z(t) = n\} = \frac{1}{2\pi i}\oint_{|w|=r} w^{Z(t)-n} \cdot \frac{dw}{w}. \tag{13}$$

Taking expectations of both sides, we get

$$\mathrm{P}_m\{Z(t) = n\} = \frac{1}{2\pi i}\oint_{|w|=r} \mathrm{E}_m\left[\left(\frac{w}{\beta}\right)^{Z(t)-n}\right]\frac{dw}{w} = \frac{1}{2\pi i}\oint_{|w|=r}\left(\frac{w}{\beta}\right)^{m-n}\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w))t}\,\frac{dw}{w}. \tag{14}$$

Now we switch from contour integration over $\mathbb{C}$ to line integration over $\mathbb{R}^2$ in polar coordinates centered at $(0, 0)$. The contour curve is a circle, so the radius r is fixed. Since the contour integral has the same value for all non-zero radii, we choose $r = 1$ or $w = e^{i\theta} = (\cos\theta, \sin\theta)$ for all $-\pi < \theta < \pi$. We now have

$$\begin{aligned} \mathrm{P}_m\{Z(t) = n\} &= \frac{\beta^{n-m}}{2\pi}\cdot\int_{-\pi}^{\pi}\cos\left((m-n)\cdot\theta\right)\cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\theta)t}\,d\theta \\ &= e^{-2\gamma\cdot\delta(\beta)t}\beta^{n-m}\cdot\frac{1}{\pi}\int_0^{\pi}\cos\left((m-n)\cdot\theta\right)\cdot e^{-2\gamma t\cdot\cos\theta}\,d\theta \\ &= e^{-2\alpha t}\beta^{n-m}\cdot I_{n-m}(2\gamma t), \end{aligned}$$

where $I_n(\cdot)$ is the n^{th} *modified Bessel function*, which is defined to be

$$I_n(x) = \sum_{m=0}^{\infty}\frac{1}{m!\cdot(m+n)!}\left(\frac{x}{2}\right)^{2m+n} = \frac{1}{\pi}\int_0^{\pi}\cos\left((m-n)\cdot\theta\right)\cdot e^{-x\cdot\cos\theta}\,d\theta. \tag{15}$$

□

As shown in Feller [14], these transition probabilities can be solved with modified Bessel functions. A multi-dimensional analogue to these functions were introduced in Massey [29] for free processes on the d dimensional integer lattice $\mathbb{Z}^d$.

3.3 Free Process Action Symmetry

Given that the cosine function is even, we have the following *group symmetry* for the free process transition probabilities:

$$\frac{\mathrm{P}_m\{Z(t) = n\}}{\beta^{n-m}} = \frac{\mathrm{P}_{\sigma(m)}\{Z(t) = \sigma(n)\}}{\beta^{\sigma(n)-\sigma(m)}}, \tag{16}$$

for all invertible, state space transformations $\sigma \in \mathcal{G}$, where

$$\mathcal{G} \equiv \{\,\sigma \,|\, \sigma(n) = n + \ell \;\text{ or }\; \sigma(n) = -n + \ell \text{ for some } \ell \in \mathbb{Z} \text{ and all } n \in \mathbb{Z}.\} \tag{17}$$

From a geometric perspective, the graphs of all the functions in $\mathcal{G}$ are lines whose set of slopes equal $\{-1, 1\}$ and whose y-intercepts are all integers in $\mathbb{Z}$. One should also notice that algebraically, $\mathcal{G}$ has the following key property:

> The invertible mappings of $\mathcal{G}$ are closed with respect to functional composition as an operation and functional inverses.

Since the composition of functions is always an associative operation, any set $\mathcal{G}$ with this property is called a *group* that *acts* on the state space. The set of slopes $\{-1, 1\}$ also form a *multiplicative* group. Moreover, the set of y-intercepts $\mathbb{Z}$ form an *additive* group. The construction that builds $\mathcal{G}$ out of these two simpler groups is called the *semi-direct product* that we denote as $\mathcal{G} = \{-1, 1\} \times_s \mathbb{Z}$. If we liken $\mathcal{G}$ to a discrete version of "Euclidean rigid motions", then $\mathcal{G}$ decomposes into a *multiplicative* group of one-dimensional "rotations" $\{1, -1\}$ and an *additive* group of integer "translations" $\mathbb{Z}$.

For modelling bike sharing systems, $\mathcal{G}$ is a set of *action symmetries* that preserve the actions or operations of the bike sharing system. A bike return action, for example, is either transformed by a given $\sigma \in \mathcal{G}$ into a rental action, with $\sigma(n+1) = \sigma(n) - 1$ for all n, or it stays a return action, where $\sigma(n+1) = \sigma(n) + 1$. Our group symmetry relationship is now equivalent to the identity

$$\beta^{\sigma^{-1}(m)-m} \cdot \mathrm{P}_{\sigma^{-1}(m)} \{Z(t) = n\} = \beta^{n-\sigma(n)} \cdot \mathrm{P}_m \{Z(t) = \sigma(n)\} \tag{18}$$

for all $\sigma \in \mathcal{G}$.

Although the state space for the free process is $\mathbb{Z}$, the state space for both the single-barrier absorbing and reflecting processes is the *positive integers* $\mathbb{Z}_+ \equiv \{0, 1, 2, \ldots\}$. Notice that all elements of $\mathbb{Z}$ are of the form n or $-n$ for some $n \in \mathbb{Z}_+$. This is the multiplicative group of slopes $\{1, -1\}$ acting on every $n \in \mathbb{Z}_+$. We can summarize this result in terms of group actions as

$$\mathbb{Z}/\{1, -1\} \cong \mathbb{Z}_+ \tag{19}$$

where we define an equivalence class that equates an integer with its own negative or additive inverse. Moreover, note that for all $n \in \mathbb{Z}$, we have $n = -n$ if and only if $n = 0$, which is our unique single-barrier state. We call $\{0\}$ the *boundary* for the state space $\mathbb{Z}_+$.

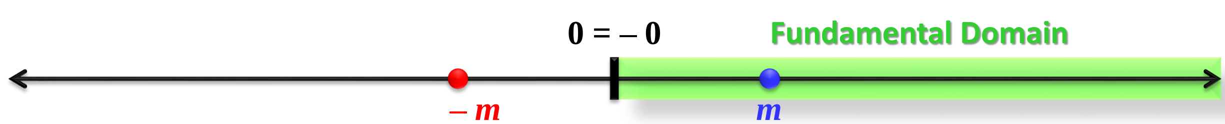

Figure 3: Single Barrier Symmetry Group: $m \to \pm m$.

This also shows us that $\mathbb{Z}_+$ is a *minimal* subset of $\mathbb{Z}$ whose group orbit reconstructs $\mathbb{Z}$. Such a subset is called a *fundamental domain* with respect to the group action $\{1, -1\}$. If we think of multiplication by -1 as a reflection about the zero state $\{0\}$, then all these results are summarized geometrically in Figure 3.

The state space for both the two-barrier absorbing and reflection processes related to $M/M/1/k$ queueing is the finite set $\{0, 1, \ldots, k\}$ where 0 and k are the two *boundary states*. Now define $\mathcal{G}_k$ to be a *subgroup* of $\mathcal{G}$, where

$$\mathcal{G}_k \equiv \{1, -1\} \times_s 2k\mathbb{Z}. \tag{20}$$

Proposition 3.2. *Our* $\boldsymbol{M/M/1/k}$ *state space is a fundamental domain with respect to the group action of* $\mathcal{G}_k$ *on* $\mathbb{Z}$ *or*

$$\mathbb{Z}/\mathcal{G}_k = (\mathbb{Z}/2k\mathbb{Z})\,/\{-1, 1\} \cong \{0, 1, \ldots, k\}. \tag{21}$$

Also note that for all integers n, we have $n = -n$ if and only if $n = 0$ and $n = 2k - n$ if and only if $n = k$.

Proof. This follows from the argument

$$(\mathbb{Z}/2k\mathbb{Z})\,/\{-1, 1\} \cong \{-k+1, \ldots, -1, 0, 1, \ldots, k\}/\{-1, 1\} \cong \{0, 1, \ldots, k\}. \tag{22}$$

□

All these symmetry results are summarized in Figures 3 and 4.

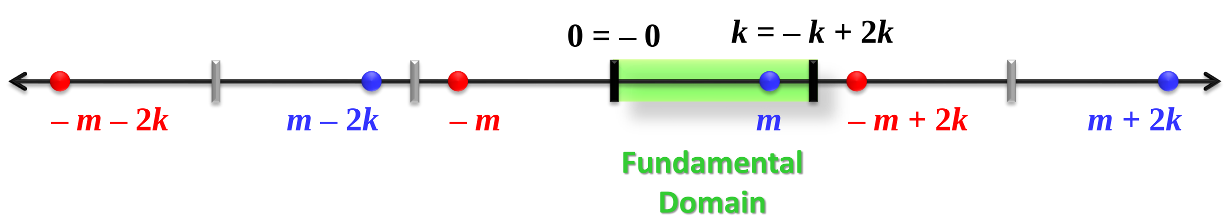

Figure 4: Double Barrier Symmetry Group: $m \to \pm m + 2k\ell$ for all $\ell \in \mathbb{Z}$.

4 Absorption Process Analysis

Now that we have given a thorough analysis of the free process, we show how to use the free process to study other quantities and performance measures of interest to related queueing systems. We can model the autonomous evolution of a bike sharing station until it becomes balanced as a stopped version of both the queueing and free processes $\{Q^*_k(t)\,|\,t \geq 0\}$, where

$$Q^*_k(t) \equiv Q_k\left(T_{0,k} \wedge t\right) \quad \text{where} \quad T_{0,k} \equiv \min\left\{t\,|Q_k(t) = 0 \text{ or } k\right\}. \tag{23}$$

We call this stopped process our *absorbing process*. The transition probabilities for this absorbing process decompose as follows:

$$\mathrm{P}_m\left\{Q_k^*(t)=n\right\}=\begin{cases}\mathrm{P}_m\left\{Q_k(t)=n, T_{0,k}>t\right\} & \text{when } 0<m<k \text{ and } 0<n<k, \\ 0 & \text{when } m=0 \text{ or } m=k \text{ but } m\neq n, \\ \mathrm{P}_m\left\{Q_k(T_{0,k})=n, T_{0,k}\leq t\right\} & \text{when } m\neq n \text{ but } n=0 \text{ or } n=k, \\ 1 & \text{when } m=n=0 \text{ or } m=n=k.\end{cases} \tag{24}$$

The first case gives the transition probabilities for the transient states. The third case gives the transitions from the transient states to the absorbing states. These are also referred to as *exit probabilities*. We provide a visualization of the absorbing process in Figure 5 where we see that there are no arrows that flow outward of the states $\{0,k\}$.

4.1 Absorption Process Forward and Backward Equations

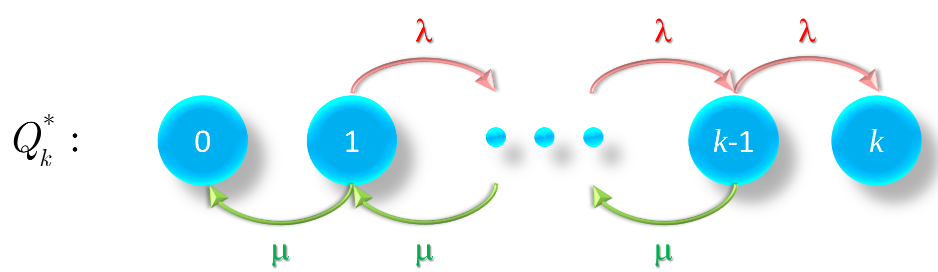

Figure 5: State Transition Diagram for the Absorbing Process with Capacity k.

Since the absorbing states for this process are the boundary states $\{0,k\}$, for all states $0<m<k$ and $0<n<k$ and $t\geq 0$, the forward equations for the absorbing process are

$$\frac{d}{dt}\mathrm{P}_m\{Q_k^*(t)=n\}=\lambda\cdot\mathrm{P}_m\{Q_k^*(t)=n-1\}+\mu\cdot\mathrm{P}_m\{Q_k^*(t)=n+1\}-(\lambda+\mu)\cdot\mathrm{P}_m\{Q_k^*(t)=n\}, \tag{25}$$

and the backward equations for the absorbing process are

$$\frac{d}{dt}\mathrm{P}_m\{Q_k^*(t)=n\}=\lambda\cdot\mathrm{P}_{m+1}\{Q_k^*(t)=n\}+\mu\cdot\mathrm{P}_{m-1}\{Q_k^*(t)=n\}-(\lambda+\mu)\cdot\mathrm{P}_m\{Q_k^*(t)=n\}, \tag{26}$$

since

$$\mathrm{P}_m\left\{Q_k(t)=0,\, T_{0,k}>t\right\}=\mathrm{P}_m\left\{Q_k(t)=k,\, T_{0,k}>t\right\}=0 \tag{27}$$

and

$$\mathrm{P}_0\left\{Q_k(t)=n,\, T_{0,k}>t\right\}=\mathrm{P}_k\left\{Q_k(t)=n,\, T_{0,k}>t\right\}=0. \tag{28}$$

4.2 Absorption Transience Using Action Symmetries

We can give an exact solution for the transition probabilities of the absorbing process in terms of the solution we have for the free process. We do this by using the group symmetry property for the free process. We begin by working out the case of $k = \infty$, which is absorbing process for the $M/M/1/\infty$ queue. The forward equations are the same are for the free process for all positive integers m and n with $m + n > 0$ and the boundary conditions

$$\mathrm{P}_m \left\{ Q_\infty(t) = 0,\, T_0 > t \right\} = \mathrm{P}_0 \left\{ Q_\infty(t) = n,\, T_0 > t \right\} = 0 \tag{29}$$

where $T_0 \equiv T_{(0,\infty)}$.

Proposition 4.1. *For all positive integers m and n we have*

$$\mathrm{P}_m \left\{ Q^*_\infty(t) = n \right\} = \mathrm{P}_m \left\{ Z(t) = n \right\} - \beta^{-2m} \cdot \mathrm{P}_{-m} \left\{ Z(t) = n \right\} \tag{30}$$

and

$$\mathrm{P}_m \left\{ Q^*_\infty(t) = n \right\} = \mathrm{P}_m \left\{ Z(t) = n \right\} - \beta^{2n} \cdot \mathrm{P}_m \left\{ Z(t) = -n \right\}. \tag{31}$$

Proof. Using our group symmetry properties, we can verify these formulas in 3 steps:

1. *Show that these expressions solve the same set of forward equations for the free process.*

   This follows from the linearity of these differential equations and Equation (30).

2. *Show that when only the terminal value equals* 0*, then the expression equals zero.*

   As a special case of Equation (31). we have

   $$\mathrm{P}_m \left\{ Q^*_\infty(t) = 0 \right\} = \mathrm{P}_m \left\{ Z(t) = 0 \right\} - \mathrm{P}_m \left\{ Z(t) = 0 \right\} = 0 \tag{32}$$

3. *Show that when $t = 0$, the expression equals one instead of zero if and only if the initial state equals the terminal state.*

   This follows from our state space being a fundamental domain for the group multiplication action of $\{1, -1\}$. Moreover, all the interior states are fixed points only for the identity mapping of multiplication by 1.

□

After observing that $0 = n - n$ and $2n = n - (-n)$, we can apply these arguments to the $M/M/1/k$ queue absorbing process. Below is a simplified version of an argument given for closed cyclic Jackson networks in Baccelli et al [3].

Proposition 4.2. *For all integers $0 < m < k$ and $0 < n < k$ we have*

$$\mathrm{P}_m \left\{ Q^*_k(t) = n \right\} = \sum_{\sigma \in \mathcal{G}_k} \mathrm{sgn}(\sigma) \cdot \beta^{\sigma(m) - m} \cdot \mathrm{P}_{\sigma(m)} \left\{ Z(t) = n \right\} \tag{33}$$

and

$$\mathrm{P}_m \left\{ Q^*_k(t) = n \right\} = \sum_{\sigma \in \mathcal{G}_k} \mathrm{sgn}(\sigma) \cdot \beta^{n - \sigma(n)} \cdot \mathrm{P}_m \left\{ Z(t) = \sigma(n) \right\}. \tag{34}$$

Proof. Using our group symmetry properties, we show this by proving that

$$\mathrm{P}_m\left\{Q_k^*(t)=n\right\} = \sum_{\ell=-\infty}^{\infty} \beta^{2k\ell}\cdot \mathrm{P}_{m+2k\ell}\left\{Z(t)=n\right\} - \beta^{-2m+2k\ell}\cdot \mathrm{P}_{-m+2k\ell}\left\{Z(t)=n\right\} \tag{35}$$

and

$$\mathrm{P}_m\left\{Q_k^*(t)=n\right\} = \sum_{\ell=-\infty}^{\infty} \beta^{-2k\ell}\cdot \mathrm{P}_{m}\left\{Z(t)=n+2k\ell\right\} - \beta^{2n-2k\ell}\cdot \mathrm{P}_{m}\left\{Z(t)=-n+2k\ell\right\}. \tag{36}$$

We can verify these formulas in 3 steps:

1. *Show that these expressions solve the same set of forward equations for the free process.*

   This follows from the linearity of these differential equations and Equation (33).

2. *Show that when only the terminal value equals* 0 *or* k*, then the expression equals zero.*

   As special cases of Equation (34). we have

$$\begin{aligned}
\mathrm{P}_m\left\{Q_k^*(t)=0\right\} &= \sum_{\sigma\in\mathcal{G}_k} \mathrm{sgn}(\sigma)\cdot\beta^{-2k\ell}\cdot \mathrm{P}_m\left\{Z(t)=\sigma(0)\right\} && (37)\\
&= \sum_{\ell=-\infty}^{\infty} \beta^{-2k\ell}\cdot \mathrm{P}_m\left\{Z(t)=2k\ell\right\} - \sum_{\ell=-\infty}^{\infty} \beta^{-2k\ell}\cdot \mathrm{P}_m\left\{Z(t)=2k\ell\right\} && (38)\\
&= 0 && (39)
\end{aligned}$$

   and

$$\begin{aligned}
\mathrm{P}_m\left\{Q_k^*(t)=k\right\} &= \sum_{\sigma\in\mathcal{G}_k} \mathrm{sgn}(\sigma)\cdot\beta^{k-\sigma(k)}\cdot \mathrm{P}_m\left\{Z(t)=\sigma(k)\right\} && (40)\\
&= \sum_{\ell=-\infty}^{\infty} \beta^{-2k\ell}\cdot \mathrm{P}_m\left\{Z(t)=k+2k\ell\right\} && (41)\\
&\quad - \sum_{\ell=-\infty}^{\infty} \beta^{-2k(\ell-1)}\cdot \mathrm{P}_m\left\{Z(t)=k+2k\cdot(\ell-1)\right\} &&\\
&= 0. && (42)
\end{aligned}$$

3. *Show that when* $t=0$*, the expression equals one instead of zero if and only if the initial state equals the terminal state.*

   This follows from our state space being a fundamental domain for the group action of $\mathcal{G}_k$. Moreover, all the interior states are fixed points only for the identity mapping.

□

4.3 Absorption Transience Using Contour Integration

Now we transform the solution for the transition probabilities for the absorbing process from an infinite sum over the transition probabilities for the free process to a finite sum of exponential and trigonometric terms.

Theorem 4.3. *For all positive integers m and n we have*

$$\mathrm{P}_m\left\{Q_k^*(t)=n\right\}=\frac{2\beta^{n-m}}{k}\cdot\sum_{\ell=1}^{k-1}\sin\frac{\ell m\pi}{k}\cdot\sin\frac{\ell n\pi}{k}\cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi\ell}{k}\right)t}. \tag{43}$$

This is a special case of higher dimensional closed cyclic networks as found in Baccelli et al. [3].

Proof. We have already defined α, β, γ, and $\delta(w)$. Now we introduce the *anti-symmetric* function $\epsilon_n(w)$, where[2]

$$\epsilon_n(w)\equiv\frac{w^n-w^{-n}}{2},\quad \epsilon_n(w)=-\epsilon_n(w^{-1})\ \text{ and }\ \epsilon_n(e^{i\theta})=-i\cdot\sin n\theta \tag{44}$$

for all $-\pi<\theta\leq\pi$ and $w\in\mathbb{C}$.

Combining Equation (34) or (36) with the contour integral representations for the free process transition probabilities gives us

$$\begin{aligned}\mathrm{P}_m\left\{Q_k^*(t)=n\right\} &= \sum_{\ell=-\infty}^{\infty}\beta^{-2k\ell}\cdot\left(\mathrm{P}_m\left\{Z(t)=n+2k\ell\right\}-\beta^{2n}\cdot\mathrm{P}_m\left\{Z(t)=-n+2k\ell\right\}\right)\\ &= \frac{\beta^{n-m}}{2\pi i}\cdot\sum_{\ell=-\infty}^{\infty}\oint_{|w|=r(\ell)}\left(w^{m-2k\ell-n}-w^{m-2k\ell+n}\right)\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w))t}\,\frac{dw}{w}\\ &= \frac{\beta^{n-m}}{\pi i}\cdot\sum_{\ell=-\infty}^{\infty}\oint_{|w|=r(\ell)}w^{m-2k\ell}\cdot\epsilon_n\left(w^{-1}\right)\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w)t)}\,\frac{dw}{w}\end{aligned}$$

Since we are free to choose any radius value $r(\ell)>0$ for the contour integral indexed by the integer ℓ, we select two values $r_+>1$ and $r_-<1$ that allow us the pass their respective summations inside of their corresponding contour integrals. Using this technique allows us

[2] The generalization of this function in Baccelli et al. [3] uses δ instead of ϵ.

to collapse this infinite sum into a finite sum of contour integrals.

$$
\begin{aligned}
\mathrm{P}_m\left\{Q_k^*(t)=n\right\} &= \frac{\beta^{n-m}}{\pi i}\oint_{|w|=r_+>1} w^m\cdot\sum_{\ell=0}^{\infty} w^{-2k\ell}\cdot\epsilon_n(w^{-1})\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w)t)}\,\frac{dw}{w}\\
&\quad+\frac{\beta^{n-m}}{\pi i}\oint_{|w|=r_-<1} w^m\cdot\sum_{\ell=-\infty}^{-1} w^{-2k\ell}\cdot\epsilon_n(w^{-1})\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w)t)}\,\frac{dw}{w}\\
&= \frac{\beta^{n-m}}{\pi i}\oint_{|w|=r_+>1}\frac{w^{m+2k}\cdot\epsilon_n(w^{-1})\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w))t}}{w^{2k}-1}\,\frac{dw}{w}\\
&\quad-\frac{\beta^{n-m}}{\pi i}\oint_{|w|=r_-<1}\frac{w^{m+2k}\cdot\epsilon_n(w^{-1})\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w))t}}{w^{2k}-1}\,\frac{dw}{w}\\
&= \frac{\beta^{n-m}}{k}\cdot\sum_{\ell=0}^{2k-1}\frac{1}{2\pi i}\oint_{|w-w_{2k}^{\ell}|=\eta} w^m\cdot\epsilon_n(w^{-1})\cdot e^{-2\gamma\cdot(\delta(\beta)-\delta(w))t}\cdot\frac{2k\cdot w^{2k-1}}{w^{2k}-1}\,dw,
\end{aligned}
$$

where

$$w_{2k}\equiv e^{2\pi i/2k}=e^{\pi i/k} \tag{45}$$

and all $2k$ new contours are circles of some sufficiently small radii $\eta>0$ uniquely centered about each $2k$-th root of unity. Figure 6 reveals the underlying geometry and topology of the technique. The difference of the red contour integrals is equivalent to integrating counter-clockwise around the radius r_+ circle centered at zero *minus* the integrating counter-clockwise around the radius r_- circle. In effect, we are integrating *clockwise* around the circle of radius r_-. The union of these oriented circles is the boundary of the annulus $\{\,w\,|\,r_-<|w|<r_+\,\}$. We then deform this boundary to the $2k$ radius η blue circles about the roots of unity. The resulting residue is the same as long as the deformations do not cross any singularities.

The rest follows from using the logarithmic derivative to show that

$$w^{2k}-1=\prod_{\ell=-k+1}^{k}(w-w_{2k}^{\ell}) \iff \frac{2k\cdot w^{2k-1}}{w^{2k}-1}=\frac{d}{dw}\log(w^{2k}-1)=\sum_{\ell=-k+1}^{k}\frac{1}{w-w_{2k}^{\ell}}. \tag{46}$$

Now we make repeated use of the Cauchy integral formula around the smaller contour circles about the $2k$ roots of unity. Finally, we have

$$\mathrm{P}_m\left\{Q_k^*(t)=n\right\}=\frac{\beta^{n-m}}{k}\cdot\sum_{\ell=-k}^{k-1}(w_{2k}^{\ell})^m\cdot\epsilon_n(w_{2k}^{-\ell})\cdot e^{-2\gamma\cdot\left(\delta(\beta)-\delta(w_{2k}^{\ell})\right)t} \tag{47}$$

$$=\frac{2\beta^{n-m}}{k}\cdot\sum_{\ell=1}^{k-1}\epsilon_m(w_{2k}^{\ell})\cdot\epsilon_n(w_{2k}^{-\ell})\cdot e^{-2\gamma\cdot\left(\delta(\beta)-\delta(w_{2k}^{\ell})\right)t} \tag{48}$$

$$=\frac{2\beta^{n-m}}{k}\cdot\sum_{\ell=1}^{k-1}\sin\frac{\ell m\pi}{k}\cdot\sin\frac{\ell n\pi}{k}\cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi\ell}{k}\right)t}. \tag{49}$$

□

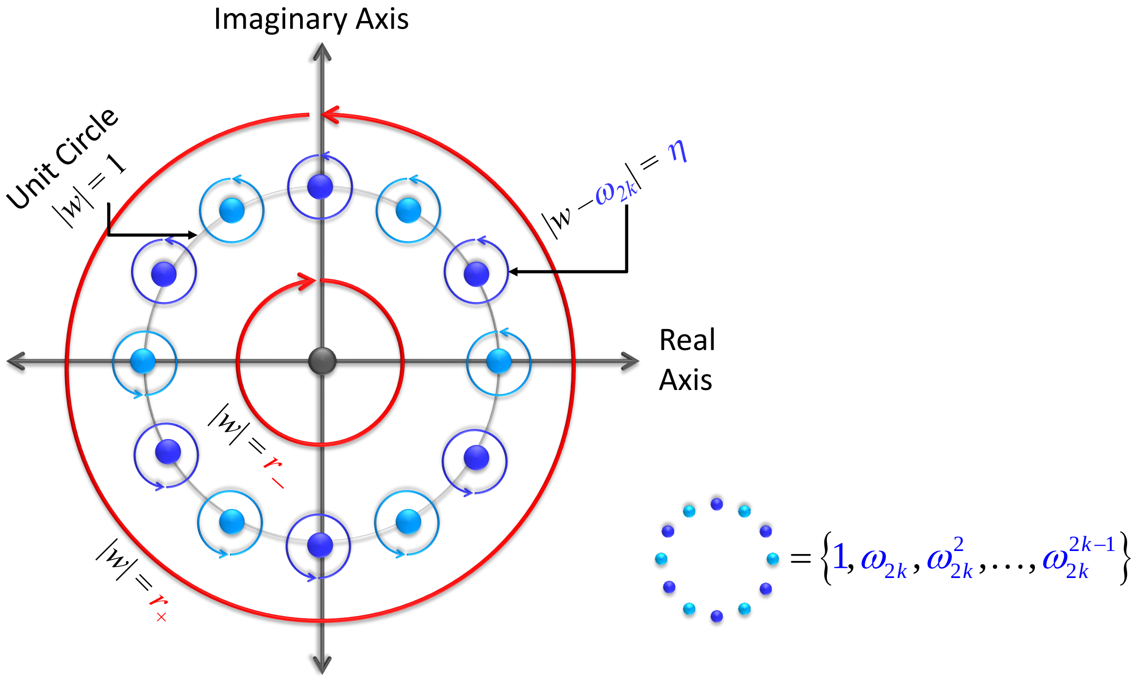

Figure 6: Contour Integrals and their Resulting Residues.

4.4 Absorbing States and Times

Now we give an exact formula for the probability of staying balanced at time t.

Theorem 4.4. *For all positive integers m, the probability of the time to rebalance exceeding time t equals*

$$\mathrm{P}_m\left\{T_{0,k} > t\right\} = \frac{\beta^{-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \left(\sin\frac{\ell\pi}{k} + \beta^k \cdot \sin\frac{(k-1)\ell\pi}{k}\right)}{\delta(\beta) - \cos\frac{\ell\pi}{k}} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t}. \tag{50}$$

Moreover, we have

$$\mathrm{E}_m\left[T_{0,k} \wedge t\right] = \frac{\beta^{-m}}{2k\gamma} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \left(\sin\frac{\ell\pi}{k} + \beta^k \cdot \sin\frac{(k-1)\ell\pi}{k}\right)}{\left(\delta(\beta) - \cos\frac{\ell\pi}{k}\right)^2} \cdot \left(1 - e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t}\right), \tag{51}$$

which yields a formula for the mean time to rebalance, *which is*

$$\mathrm{E}_m\left[T_{0,k}\right] = \frac{\beta^{-m}}{2k\gamma} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \left(\sin\frac{\ell\pi}{k} + \beta^k \cdot \sin\frac{(k-1)\ell\pi}{k}\right)}{\left(\delta(\beta) - \cos\frac{\ell\pi}{k}\right)^2}. \tag{52}$$

Proof. Our absorbing time $T_{0,k}$ for the $M/M/1/k$ queue corresponds to the *time of balance* for a bike sharing station. Thus the event $\{T_{0,k} > t\}$ is equivalent to the stopped queueing

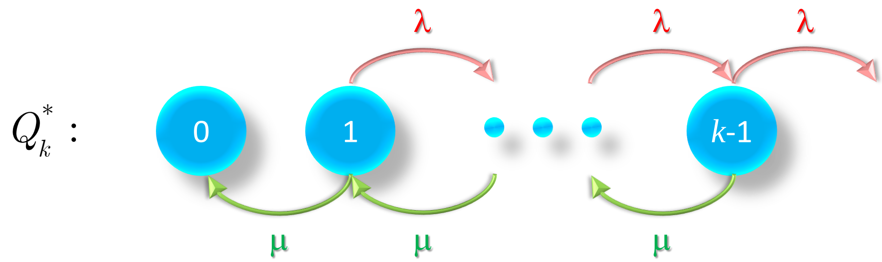

Figure 7: State Transition Diagram for the Sub-Markovian Absorbing Process with State k Concealed.

process belonging to one of the *balanced* or *transient* states found in the set $\{1, \ldots, k-1\}$. Figure 8 identifies the rate of flow out of the set of balanced states. This equals the sum of the bike return rate times the probability that the station has $k-1$ bikes plus the bike rental rate times the probability that the station has only 1 bike. This gives us

$$-\frac{d}{dt}\mathrm{P}_m\left\{T_{0,k} > t\right\} = \lambda \cdot \mathrm{P}_m\left\{Q_k^*(t) = k-1\right\} + \mu \cdot \mathrm{P}_m\left\{Q_k^*(t) = 1\right\}. \tag{53}$$

Now when we integrate from t to ∞, we obtain

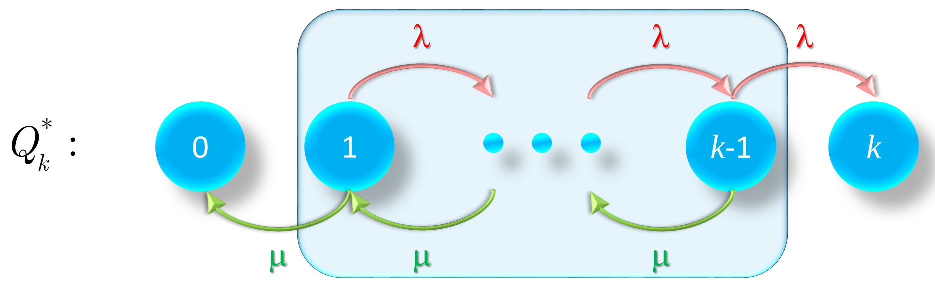

Figure 8: Net Flow for the Absorbing Process out of the Transient States.

$$\mathrm{P}_m\left\{T_{0,k} > t\right\} = \lambda \cdot \int_t^\infty \mathrm{P}_m\left\{Q_k^*(s) = k-1\right\} ds + \mu \cdot \int_t^\infty \mathrm{P}_m\left\{Q_k^*(s) = 1\right\} ds \tag{54}$$

$$= \frac{\lambda\beta^{k-1-m}}{\gamma k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \sin\frac{(k-1)\ell\pi}{k}}{\delta(\beta) - \cos\frac{\pi\ell}{k}} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t} \tag{55}$$

$$+ \frac{\mu\beta^{1-m}}{\gamma k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \sin\frac{\ell\pi}{k}}{\delta(\beta) - \cos\frac{\ell\pi}{k}} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi\ell}{k}\right)t} \tag{56}$$

$$= \frac{1}{k\beta^m} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \left(\sin\frac{\ell\pi}{k} + \beta^k \cdot \sin\frac{(k-1)\ell\pi}{k}\right)}{\delta(\beta) - \cos\frac{\ell\pi}{k}} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t}. \tag{57}$$

Setting $t = 0$, gives us the identity

$$\frac{1}{k\beta^m} \cdot \sum_{\ell=1}^{k-1} \frac{\sin \frac{\ell m \pi}{k} \cdot \left(\sin \frac{\ell \pi}{k} + \beta^k \cdot \sin \frac{(k-1)\ell\pi}{k} \right)}{\delta(\beta) - \cos \frac{\ell\pi}{k}} = 1. \tag{58}$$

Integrating over time from 0 to t, gives us the mean time to rebalance before time t. Taking the limit of $t \to \infty$ gives us a formula for the mean time to rebalance. □

Now we derive the probability distribution for being unbalanced i.e. the queue is absorbed into the empty state $\{0\}$ or the full state $\{k\}$.

Proposition 4.5. *For all integers $0 < m < k$, we have*

$$\mathrm{P}_m \left\{ Q_k \left(T_{0,k} \right) = k \right\} = \frac{\beta^{k-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin \frac{\ell m \pi}{k} \cdot \sin \frac{(k-1)\ell\pi}{k}}{\delta(\beta) - \cos \frac{\ell\pi}{k}} \tag{59}$$

and

$$\mathrm{P}_m \left\{ Q_k \left(T_{0,k} \right) = 0 \right\} = \frac{\beta^{-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin \frac{\ell m \pi}{k} \cdot \sin \frac{\ell\pi}{k}}{\delta(\beta) - \cos \frac{\ell\pi}{k}}. \tag{60}$$

Proof. We have

$$\frac{d}{dt} \mathrm{P}_m \left\{ Q_k^*(t) = 0 \right\} = \mu \cdot \mathrm{P}_m \left\{ Q_k^*(t) = 1 \right\} = \frac{2\gamma}{k\beta^m} \cdot \sum_{\ell=1}^{k-1} \sin \frac{\ell m \pi}{k} \cdot \sin \frac{\ell\pi}{k} \cdot e^{-2\gamma \cdot \left(\delta(\beta) - \cos \frac{\ell\pi}{k} \right) t}. \tag{61}$$

A similar argument gives us

$$\frac{d}{dt} \mathrm{P}_m \left\{ Q_k^*(t) = k \right\} = \frac{2\gamma\beta^k}{k\beta^m} \cdot \sum_{\ell=1}^{k-1} \sin \frac{\ell m \pi}{k} \cdot \sin \frac{(k-1)\ell\pi}{k} \cdot e^{-2\gamma \cdot \left(\delta(\beta) - \cos \frac{\ell\pi}{k} \right) t}. \tag{62}$$

Integrating over the positive reals gives us the exit probability from the transient states to either the absorbing full state or the absorbing empty state. □

By definition, the $M/M/1/k$ queueing process before the time of rebalance behaves just like the free process. A special case of a martingale argument that was used for higher dimensional networks in Baccelli and Massey Baccelli and Massey [2] gives us the following expression

$$\mathrm{E}_m \left[Q_k^*(t) \right] = m + (\lambda - \mu) \cdot \mathrm{E}_m \left[T_{0,k} \wedge t \right]. \tag{63}$$

4.5 Quasi-Steady State Absorption Distributions and Means

Now we study the long term state of the system, *given* that the rare event that absorption has *not* yet happened.

Theorem 4.6. *The following limiting distribution exists,*

$$p_*(n) \equiv \lim_{t\to\infty} \mathrm{P}_m \left\{ Q_k^*(t) = n \mid T_{0,k} > t \right\} = \frac{2\beta^n \cdot \left(\delta(\beta) - \cos\frac{\pi}{k}\right) \cdot \sin\frac{n\pi}{k}}{(1+\beta^k)\cdot \sin\frac{\pi}{k}}. \tag{64}$$

Moreover, initializing with this distribution gives us

$$\mathrm{P}_* \left\{ Q_k^*(t) = n \right\} = p_*(n) \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi}{k}\right)t} \quad \text{and} \quad \mathrm{P}_* \left\{ T_{0,k} > t \right\} = e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi}{k}\right)t} \tag{65}$$

for all $t > 0$, where

$$\mathrm{E}_* T_{0,k} = \frac{1}{2\gamma \cdot \left(\delta(\beta) - \cos\frac{\pi}{k}\right)}. \tag{66}$$

Finally, we have

$$\mathrm{E}_* \left[Q_k^*(t) \right] = \frac{k\beta^k}{1+\beta^k} - \frac{2\,\epsilon_1(\beta)\cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\pi}{k})t}}{\delta(\beta)-\cos\frac{\pi}{k}}. \tag{67}$$

This limit is called the *quasi-steady state* distribution for the transient states of the $M/M/1/k$ queue with respect to the time to rebalance $T_{0,k}$.

Proof. For all $0 < n < k$, we now have

$$\begin{aligned}
\mathrm{P}_* \left\{ Q_k^*(t) = n \right\} &= \sum_{m=1}^{k-1} p_*(m) \cdot \mathrm{P}_m \left\{ Q_k^*(t) = n \right\} \\
&= \sum_{m=1}^{k-1} p_*(m) \cdot \frac{2\beta^{n-m}}{k} \cdot \sum_{\ell=1}^{k-1} \sin\frac{\ell m\pi}{k} \cdot \sin\frac{\ell n\pi}{k} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi\ell}{k}\right)t} \\
&= \frac{p_*(m)}{\sin\frac{m\pi}{k}} \cdot \sum_{\ell=1}^{k-1} \sin\frac{\ell n\pi}{k} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi\ell}{k}\right)t} \cdot \frac{2}{k} \cdot \sum_{m=1}^{k-1} \sin\frac{m\pi}{k} \sin\frac{\ell m\pi}{k}.
\end{aligned}$$

This gives us the quasi-stationarity result since the ratio $p_*(m)/\sin\frac{m\pi}{k}$ is independent of m and

$$\mathrm{P}_\ell \left\{ Q_k^*(0) = 1 \right\} = \frac{2}{k} \cdot \sum_{m=1}^{k-1} \sin\frac{m\pi}{k} \sin\frac{\ell m\pi}{k} = \begin{cases} 1 & \text{when } \ell = 1, \\ 0 & \text{when } \ell \neq 1, \end{cases} \tag{68}$$

Moreover, since

$$\mathrm{E}_* \left[Q_k^*(t) \right] = \mathrm{E}_* \left[Q_k^*(t), T_{0,k} > t \right] + \mathrm{E}_* \left[Q_k^* \left(T_{0,k} \right), T_{0,k} \le t \right], \tag{69}$$

we then have

$$\begin{aligned}
\mathrm{E}_*[Q_k^*(t)] &= \sum_{n=1}^{k-1} n \cdot \mathrm{P}_* \left\{ Q_k^*(t) = n \right\} + k\lambda \cdot \int_0^t \mathrm{P}_* \left\{ Q_k^*(s) = k-1 \right\} ds \\
&= \sum_{n=1}^{k-1} n \cdot p_*(n) \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\pi}{k})t} + k\lambda \cdot p_*(k-1) \cdot \int_0^t e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\pi}{k})s}\, ds \\
&= \frac{2\left(\delta(\beta)-\cos\frac{\pi}{k}\right)}{(1+\beta^k)\cdot\sin\frac{\pi}{k}} \cdot \left(\sum_{n=1}^{k-1} n \cdot \beta^n \cdot \sin\frac{n\pi}{k} \right) \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\pi}{k})t} \\
&\quad + \frac{k\beta^k}{1+\beta^k} \cdot \left(1 - e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\pi}{k})t}\right).
\end{aligned}$$

Since the quasi-steady state distribution sums to one, we have the identity

$$\sum_{m=1}^{k-1} \beta^m \cdot \sin \frac{m\pi}{k} = \frac{(1+\beta^k) \cdot \sin \frac{\pi}{k}}{2\left(\delta(\beta) - \cos \frac{\pi}{k}\right)}. \tag{70}$$

Differentiating with respect to β and then multiplying by β gives us

$$\sum_{m=1}^{k-1} m\beta^m \cdot \sin \frac{m\pi}{k} = \frac{(1+\beta^k) \cdot \sin \frac{\pi}{k}}{2\left(\delta(\beta) - \cos \frac{\pi}{k}\right)} \cdot \left(\frac{k\beta^k}{1+\beta^k} - \frac{\beta - 1/\beta}{\delta(\beta) - \cos \frac{\pi}{k}} \right). \tag{71}$$

Finally, we have

$$\mathrm{E}_* \left[Q_k^*(0)\right] = \frac{2\left(\delta(\beta) - \cos \frac{\pi}{k}\right)}{(1+\beta^k) \cdot \sin \frac{\pi}{k}} \cdot \left(\sum_{m=1}^{k-1} m\beta^m \cdot \sin \frac{m\pi}{k} \right) = \frac{k\beta^k}{1+\beta^k} - \frac{\beta - 1/\beta}{\delta(\beta) - \cos \frac{\pi}{k}} \tag{72}$$

and

$$\mathrm{E}_* \left[Q_k^*(t)\right] = \mathrm{E}_* \left[Q_k^*(0)\right] \cdot e^{-2\gamma \cdot (\delta(\beta) - \cos \frac{\pi}{k})t} + \frac{k\beta^k}{1+\beta^k} \cdot \left(1 - e^{-2\gamma \cdot (\delta(\beta) - \cos \frac{\pi}{k})t}\right) \tag{73}$$

$$= \left(\frac{k\beta^k}{1+\beta^k} - \frac{\beta - 1/\beta}{\delta(\beta) - \cos \frac{\pi}{k}} \right) \cdot e^{-2\gamma(\delta(\beta) - \cos \frac{\pi}{k})t} + \frac{k\beta^k}{1+\beta^k} \cdot \left(1 - e^{-2\gamma \cdot (\delta(\beta) - \cos \frac{\pi}{k})t}\right) \tag{74}$$

$$= \frac{k\beta^k}{1+\beta^k} - \frac{2\,\epsilon_1(\beta) \cdot e^{-2\gamma \cdot (\delta(\beta) - \cos \frac{\pi}{k})t}}{\delta(\beta) - \cos \frac{\pi}{k}}. \tag{75}$$

This completes the proof. □

5 Reflection Process Analysis

In this section, we exploit the free process and absorbed process analysis of the previous sections to develop new results for the doubly reflected $M/M/1/k$ queue. In Figure 9, we provide a state transition diagram for the $M/M/1/k$ queue.

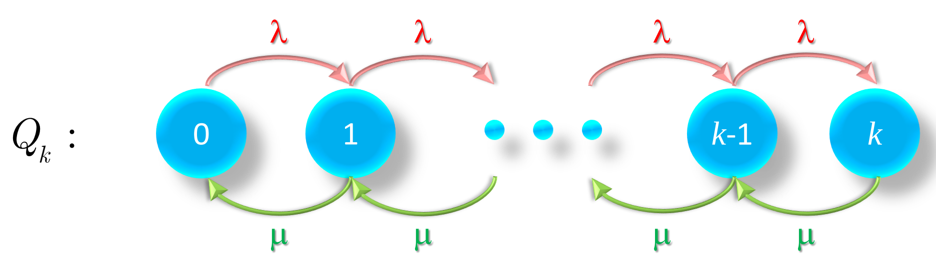

Figure 9: State Transition Diagram for the Reflecting Process with Capacity k.

5.1 Reflection Process Forward and Backward Equations

First, we start off with the Kolmogorov forward equations for the states of the $M/M/1/k$ queue. The Kolmogorov forward equations, for all states $0 < n < k$, are

$$\begin{aligned}\frac{d}{dt}\mathrm{P}_m\left\{Q_k(t)=n\right\} = \lambda\cdot\mathrm{P}_m\{Q_k(t)=n-1\}+\mu\cdot\mathrm{P}_m\{Q_k(t)=n+1\}\\ -(\lambda+\mu)\cdot\mathrm{P}_m\{Q_k(t)=n\}\,.\end{aligned} \tag{76}$$

Moreover, on the boundary we have that

$$\frac{d}{dt}\mathrm{P}_m\{Q_k(t)=k\} = \lambda\cdot\mathrm{P}_m\{Q_k(t)=k-1\}-\mu\cdot\mathrm{P}_m\{Q_k(t)=k\}\,, \tag{77}$$

and

$$\frac{d}{dt}\mathrm{P}_m\left\{Q_k(t)=0\right\} = \mu\cdot\mathrm{P}_m\left\{Q_k(t)=1\right\}-\lambda\cdot\mathrm{P}_m\left\{Q_k(t)=0\right\}. \tag{78}$$

5.2 Action Expansion for a Markov Queueing Generator

We can give a matrix summary for the forward and backward equations as

$$\frac{d}{dt}\mathbf{P}(t)=\mathbf{P}(t)\cdot\mathbf{A}=\mathbf{A}\cdot\mathbf{P}(t). \tag{79}$$

with $k+1$-dimensional operators or square matrices

$$\mathbf{P}(t)=\begin{bmatrix}\mathrm{P}_0\{Q_k(t)=0\} & \mathrm{P}_0\{Q_k(t)=1\} & \cdots & \mathrm{P}_0\{Q_k(t)=k\}\\ \mathrm{P}_1\{Q_k(t)=0\} & \mathrm{P}_1\{Q_k(t)=1\} & \cdots & \mathrm{P}_1\{Q_k(t)=k\}\\ \vdots & \vdots & \ddots & \vdots\\ \mathrm{P}_k\{Q_k(t)=0\} & \mathrm{P}_k\{Q_k(t)=1\} & \cdots & \mathrm{P}_k\{Q_k(t)=k\}\end{bmatrix} \tag{80}$$

and

$$\mathbf{A}=\begin{bmatrix}-\lambda & \lambda & 0 & \cdots & 0 & 0\\ \mu & -(\lambda+\mu) & \lambda & \ddots & 0 & 0\\ 0 & \mu & -(\lambda+\mu) & \ddots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & -(\lambda+\mu) & \lambda\\ 0 & 0 & 0 & \cdots & \mu & -\mu\end{bmatrix} \tag{81}$$

We call $\mathbf{A}$ the *Markov generator* for the Markovian $M/M/1/k$ queueing process. We can express this generator more compactly in terms of more fundamental operators, namely

$$\mathbf{A}=\lambda\mathbf{R}+\mu\mathbf{L}-\lambda\mathbf{R}\mathbf{L}-\mu\mathbf{L}\mathbf{R}. \tag{82}$$

We call this the *action expansion* for $\mathbf{A}$, where $\mathbf{R}$ is the *right-shift operator* on row vectors with

$$\mathbf{R} \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \Longleftrightarrow \quad \mathbf{e}_j\mathbf{R} = \begin{cases} \mathbf{e}_{j+1} & \text{when } j = 0, 1, \ldots, k-1, \\ 0 & \text{when } j = k, \end{cases} \tag{83}$$

and $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is defined to be the set of *unit basis vectors* for the states $\{0, 1, \ldots, k\}$ respectively. Note that they form an orthonormal basis for a $k+1$ dimensional vector space.

Similarly, we define $\mathbf{L}$ to be the *left-shift operator* where

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad \Longleftrightarrow \quad \mathbf{e}_j\mathbf{L} = \begin{cases} 0 & \text{when } j = 0, \\ \mathbf{e}_{j-1} & \text{when } j = 1, 2, \ldots, k. \end{cases} \tag{84}$$

The operators $\mathbf{R}$ and $\mathbf{L}$ have the following set of algebraic relations

$$\mathbf{L}^{k+1} = \mathbf{R}^{k+1} = \mathbf{0},\ \mathbf{LRL} = \mathbf{L},\ \mathbf{RLR} = \mathbf{R},\ \text{and } \mathbf{L}^{\mathsf{T}} = \mathbf{R} \tag{85}$$

We call $\mathbf{R}$ and $\mathbf{L}$ *action operators*. In terms of how they act on unit basis vectors, they encode our $M/M/1/k$ transitions for bike returns (arrivals) and bike rentals (departures) respectively. Observe that both products $\mathbf{LR}$ and $\mathbf{RL}$ are both projection operators, since $(\mathbf{LR})^{\mathbf{2}} = \mathbf{LR}$ and $(\mathbf{RL})^{\mathbf{2}} = \mathbf{RL}$. The projection operator $\mathbf{LR}$ validates all states that are non-empty. These are the same states where a service departure is possible and $\mathbf{e}_0\mathbf{LR} = 0$. Similarly, $\mathbf{RL}$ validates all the non-full states, which excludes state k and is equivalent to having $\mathbf{e}_k\mathbf{RL} = 0$.

5.3 Spectral Symmetry for a sub-Markov Generator

For our specific Markov generator, we use its action expansion to construct an invertible transformation where the resulting matrix is a sub-Markov generator and it has the same set of eigenvalues. We call this new matrix the *spectral conjugate* of the original Markov generator.

Lemma 5.1. *The $M/M/1/k$ Markov generator has a spectral conjugate $\widehat{\mathbf{A}}$ where*

$$\widehat{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \lambda & -(\lambda+\mu) & \mu & \ddots & 0 & 0 \\ 0 & \lambda & -(\lambda+\mu) & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -(\lambda+\mu) & \mu \\ 0 & 0 & 0 & \cdots & \lambda & -(\lambda+\mu) \end{bmatrix}. \tag{86}$$

Moreover, we can show that for all $0 \le m \le k$ and $0 \le n \le k$ we have

$$\mathrm{P}_m\{Q_k(t) \ge n\} = \mathrm{P}_{k+1-n}\{Q^*_{k+1}(t) \ge k+1-m\}. \tag{87}$$

The operator $\hat{\mathbf{A}}$ is a sub-Markovian generator for an $M/M/1/k+1$ absorbing process, where the roles of λ and μ are reversed. Here the return rate is μ and the rental rate is λ. The balanced or transient states here belong to the set $\{1, \ldots, k\}$, the absorbing state explicitly shown is 0, and the concealed absorbing state is $k+1$. The result is a $k+1$ dimensional square matrix for its generator. Also note that the "particle, anti-particle" dual to this process is $\{Q^*_{k+1}(t)\,|\,t \ge 0\}$, where we conceal the empty state 0.

Proof. From our action decomposition for $M/M/1/k$ generators, we have

$$\mathbf{A} = (\lambda\mathbf{R} - \mu\mathbf{LR}) \cdot (\mathbf{I} - \mathbf{L}) \quad \text{and} \quad \widehat{\mathbf{A}} \equiv \lambda\mathbf{L} + \mu\mathbf{LR^2} - (\lambda+\mu)\mathbf{LR} = (\lambda\mathbf{L} - \mu\mathbf{LR}) \cdot (\mathbf{I} - \mathbf{R}). \tag{88}$$

We now see that

$$\widehat{\mathbf{A}} = \big((\mathbf{I} - \mathbf{L}) \cdot \mathbf{A} \cdot (\mathbf{I} - \mathbf{L})^{-1}\big)^{\mathsf{T}}. \tag{89}$$

Finally note that similarity transformations and transposes are both invertible and spectral preserving.

For the second result, we construct the following operators

$$(\mathbf{I} - \mathbf{L})^{-1} = \sum_{j=0}^{\infty} \mathbf{L}^j = \sum_{j=0}^{k} \mathbf{L}^j = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \ddots & 0 & 0 \\ 1 & 1 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \ddots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \tag{90}$$

and

$$(\mathbf{I} - \mathbf{R})^{-1} = \sum_{j=0}^{\infty} \mathbf{R}^j = \sum_{j=0}^{k} \mathbf{R}^j = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \ddots & 1 & 1 \\ 0 & 0 & 0 & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{91}$$

If X is a random variable with values from our state space $\{0, 1, \ldots, k\}$, define $\mathbf{e}_X$ to be the following random vector of indicator functions

$$\mathbf{e}_X \equiv \sum_{j=0}^{k} \{X = j\}\, \mathbf{e}_j. \tag{92}$$

This gives us

$$\mathbf{e}_X \cdot (\mathbf{I} - \mathbf{L})^{-1} \equiv \sum_{j=0}^{k} \{X \geq j\}\, \mathbf{e}_j \;\text{ and }\; \mathbf{e}_X \cdot (\mathbf{I} - \mathbf{R})^{-1} \equiv \sum_{j=0}^{k} \{X \leq j\}\, \mathbf{e}_j. \tag{93}$$

If we take expectations and use the identity $\mathrm{E}\{X = j\} = \mathrm{P}\{X = j\}$ then

$$\mathrm{E}\left[\mathbf{e}_X\right] \equiv \sum_{j=0}^{k} \mathrm{P}\left\{X = j\right\} \mathbf{e}_j, \tag{94}$$

which encodes the distribution for X as a *probability vector*. Moreover, we also get

$$\mathrm{E}\left[\mathbf{e}_X\right] \cdot (\mathbf{I} - \mathbf{L})^{-1} \equiv \sum_{j=0}^{k} \mathrm{P}\left\{X \geq j\right\} \mathbf{e}_j \;\text{ and }\; \mathrm{E}\left[\mathbf{e}_X\right] \cdot (\mathbf{I} - \mathbf{R})^{-1} \equiv \sum_{j=0}^{k} \mathrm{P}\left\{X \leq j\right\} \mathbf{e}_j. \tag{95}$$

The similarity and transpose relationships between $\mathbf{A}$ and $\widehat{\mathbf{A}}$ now give us

$$\exp(t\mathbf{A}) \cdot (\mathbf{I} - \mathbf{L})^{-1} = \left(\exp(t\widehat{\mathbf{A}}) \cdot (\mathbf{I} - \mathbf{R})^{-1}\right)^{\mathsf{T}}, \tag{96}$$

This encodes the relationship between the probability distributions $Q_k(t)$ and $k+1-Q^*_{k+1}(t)$. $\square$

We now conclude this section with simple consequence of this lemma.

Corollary 5.2. *Since a tail distribution is a decreasing function of the state space, then for all* $0 \leq m < k$ *and* $0 \leq m \leq k$*, we have*

$$\mathrm{P}_m \left\{Q_k(t) \leq n\right\} \leq \mathrm{P}_{m+1} \left\{Q_k(t) \leq n\right\}. \tag{97}$$

Any process that has this specific type of a spectral conjugate is said to be *Möbius monotone*, see [30] for more details. These properties hold for all birth-death processes.

5.4 Reflection Transience Using Spectral Symmetry

Using spectral symmetry, we can derive reflection transience in terms of absorbing transience. We begin with a useful ϵ_m function identity.

Lemma 5.3. *For all complex constants* a *and* w *such that* $\delta(a) \neq \delta(w)$*, we have*

$$2\sum_{j=1}^{n} a^j \cdot \epsilon_j(w) = \frac{\epsilon_1(w) + a^n \cdot \left(a \cdot \epsilon_n(w) - \epsilon_{n+1}(w)\right)}{\delta(a) - \delta(w)}. \tag{98}$$

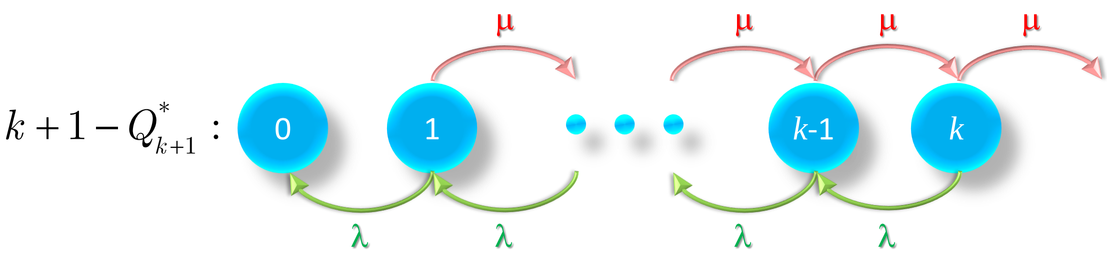

Figure 10: State Transition Diagram for the Sub-Markovian, Absorbing Conjugate with Capacity $k+1$ to the Reflecting Process with Capacity k.

Proof. Observing that $\epsilon_0(w) = 0$ and

$$\delta(w) \cdot \epsilon_n(w) = \frac{\epsilon_{n+1}(w) + \epsilon_{n-1}(w)}{2}, \tag{99}$$

it follows that

$$\begin{aligned} 2\,\delta(w) \cdot \sum_{j=1}^{n} a^j \epsilon_j(w) &= \sum_{j=1}^{n} a^j \cdot (\epsilon_{j+1}(w) + \epsilon_{j-1}(w)) \\ &= \frac{1}{a} \cdot \sum_{j=1}^{n} a^{j+1} \cdot \epsilon_{j+1}(w) + a \cdot \sum_{j=1}^{n} a^{j-1} \cdot \epsilon_{j-1}(w) \\ &= 2\,\delta(a) \cdot \sum_{j=1}^{n} a^j \cdot \epsilon_j(w) + \frac{1}{a} \cdot a^{n+1} \cdot \epsilon_{n+1}(w) - \epsilon_1(w) - a \cdot a^n \cdot \epsilon_n(w) \end{aligned}$$

and the rest follows. □

Now we give the exact solution to the transition probabilities for the $M/M/1/k$ queueing or reflecting process:

Theorem 5.4. *For all integers $0 \le m \le k$ and $0 \le n \le k$, we have*

$$\begin{aligned} \mathrm{P}_m\{Q_k(t) = n\} &= \frac{(1-\rho)\cdot\rho^n}{1-\rho^{k+1}} \\ &+ \frac{\beta^{n-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\left(\sin\frac{\ell n\pi}{k+1} - \beta\cdot\sin\frac{\ell\cdot(n+1)\pi}{k+1}\right)\cdot\left(\sin\frac{\ell m\pi}{k+1} - \beta\cdot\sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta\cdot\left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)} \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t} \end{aligned} \tag{100}$$

This is the formula given in Takács [44].

Proof. Since

$$\mathrm{P}_m\{Q_k(t) = n\} = \mathrm{P}_m\{Q_k(t) \ge n\} - \mathrm{P}_m\{Q_k(t) \ge n+1\}, \tag{101}$$

we need only show that

$$\begin{aligned}&\mathrm{P}_m\{Q_k(t)\geq n\} \\ &\quad=\frac{\rho^n-\rho^{k+1}}{1-\rho^{k+1}}+\frac{\beta^{n-m}}{k+1}\cdot\sum_{\ell=1}^{k}\frac{\sin\frac{\ell n\pi}{k+1}\cdot\left(\sin\frac{\ell m\pi}{k+1}-\beta\cdot\sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k+1}\right)}\cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t}.\end{aligned}\tag{102}$$

Using the $2k+2$ roots of unity or

$$w_{2k+2}=e^{2\pi i/(2k+2)}=e^{\pi i/(k+1)},\tag{103}$$

we can rewrite these transient tail distributions as

$$\begin{aligned}&\mathrm{P}_m\{Q_k(t)\geq n\} \\ &\quad=\frac{\rho^n-\rho^{k+1}}{1-\rho^{k+1}}+\frac{\beta^{n-m}}{k+1}\cdot\sum_{\ell=1}^{k}\frac{\epsilon_n(w_{2k+2}^{\ell})\cdot\left(\epsilon_m(w_{2k+2}^{-\ell})-\beta\cdot\epsilon_{m+1}(w_{2k+2}^{-\ell})\right)}{\beta\cdot\left(\delta(\beta)-\delta(w_{2k+2}^{\ell})\right)}\cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t}.\end{aligned}\tag{104}$$

Now we use our spectral conjugate generator $\hat{\mathbf{A}}$, which gives us

$$\begin{aligned}\mathrm{P}_m\{Q_k(t)\geq n\}&=\mathrm{P}_{k+1-n}\{Q^*_{k+1}(t)\geq k+1-m\}\\ &=\mathrm{P}_{k+1-n}\{Q^*_{k+1}(t)=k+1\}+\sum_{j=1}^{m}\mathrm{P}_{k+1-n}\{Q^*_{k+1}(t)=k+1-j\} &&(105)\\ &=\lambda\cdot\int_0^t\mathrm{P}_{k+1-n}\{Q^*_{k+1}(t)(s)=k\}\,ds+\sum_{j=1}^{m}\mathrm{P}_{k+1-n}\{Q^*_{k+1}(t)=k+1-j\}. &&(106)\end{aligned}$$

Next, we replace all the absorbing probabilities by their spectral expansions. We then use the identities

$$\epsilon_m\left(w_{2k}^{\ell}\right)=(-1)^{\ell+1}\cdot\epsilon_{k+1-m}\left(w_{2k}^{\ell}\right)\implies\epsilon_m\left(w_{2k}^{\ell}\right)\cdot\epsilon_n\left(w_{2k}^{-\ell}\right)=\epsilon_{k+1-m}\left(w_{2k}^{\ell}\right)\cdot\epsilon_{k+1-n}\left(w_{2k}^{-\ell}\right).\tag{107}$$

Combining these results, we now have

$$\begin{aligned}\mathrm{P}_m\{Q_k(t)\geq n\}&=\lambda\cdot\int_0^t\frac{2\beta^{n-1}}{k+1}\cdot\sum_{\ell=1}^{k}\epsilon_{k+1-n}(w_{2k+2}^{\ell})\cdot\epsilon_k(w_{2k+2}^{-\ell})\cdot e^{-2\gamma\cdot\left(\delta(\beta)-\delta(w_{2k+2}^{\ell})\right)s}\,ds &&(108)\\ &\quad+\sum_{j=1}^{m}\frac{2\beta^{n-j}}{k+1}\cdot\sum_{\ell=1}^{k}\epsilon_{k+1-n}(w_{2k+2}^{\ell})\cdot\epsilon_{k+1-j}(w_{2k+2}^{-\ell})\cdot e^{-2\gamma\cdot\left(\delta(\beta)-\delta(w_{2k+2}^{\ell})\right)t}\\ &=\lambda\cdot\frac{2\beta^{n-1}}{k+1}\cdot\sum_{\ell=1}^{k}\frac{\epsilon_n(w_{2k+2}^{\ell})\cdot\epsilon_1(w_{2k+2}^{-\ell})}{2\gamma\cdot\left(\delta(\beta)-\delta(w_{2k+2}^{\ell})\right)}\cdot\left(1-e^{-2\gamma\cdot\left(\delta(\beta)-\delta(w_{2k+2}^{\ell})\right)t}\right) &&(109)\\ &\quad+\frac{2\beta^{n}}{k+1}\cdot\sum_{\ell=1}^{k}\epsilon_n(w_{2k+2}^{\ell})\cdot\left(\sum_{j=1}^{m}\beta^{-j}\cdot\epsilon_j(w_{2k+2}^{-\ell})\right)\cdot e^{-2\gamma\cdot\left(\delta(\beta)-\delta(w_{2k+2}^{\ell})\right)t}.\end{aligned}$$

Applying Lemma 5.3, this simplifies to

$$\mathrm{P}_m\{Q_k(t) \geq n\} = \frac{\beta^n}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\epsilon_n(w_{2k+2}^{\ell}) \cdot \epsilon_1(w_{2k+2}^{-\ell})}{\delta(\beta) - \delta(w_{2k+2}^{\ell})} \cdot \left(1 - e^{-2\gamma \cdot \left(\delta(\beta) - \delta(w_{2k+2}^{\ell})\right)t}\right) \tag{110}$$

$$+ \frac{\beta^n}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\epsilon_n(w_{2k+2}^{\ell}) \cdot \epsilon_1(w_{2k+2}^{-\ell})}{\delta(\beta) - \delta(w_{2k+2}^{\ell})} \cdot e^{-2\gamma \cdot \left(\delta(\beta) - \delta(w_{2k+2}^{\ell})\right)t}$$

$$+ \frac{\beta^{n-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\epsilon_n(w_{2k+2}^{\ell}) \cdot \left(\beta^{-1} \cdot \epsilon_m(w_{2k+2}^{-\ell}) - \epsilon_{m+1}(w_{2k+2}^{-\ell})\right)}{\delta(\beta) - \delta(w_{2k+2}^{\ell})} \cdot e^{-2\gamma \cdot \left(\delta(\beta) - \delta(w_{2k+2}^{\ell})\right)t}$$

$$= \frac{\beta^n}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\epsilon_n(w_{2k+2}^{\ell}) \cdot \epsilon_1(w_{2k+2}^{-\ell})}{\delta(\beta) - \delta(w_{2k+2}^{\ell})} \tag{111}$$

$$+ \frac{\beta^{n-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\epsilon_n(w_{2k+2}^{\ell}) \cdot \left(\beta^{-1} \cdot \epsilon_m(w_{2k+2}^{-\ell}) - \epsilon_{m+1}(w_{2k+2}^{-\ell})\right)}{\delta(\beta) - \delta(w_{2k+2}^{\ell})} \cdot e^{-2\gamma \cdot \left(\delta(\beta) - \delta(w_{2k+2}^{\ell})\right)t}.$$

Since we know the steady state distribution for the $M/M/1/k$ queue, then we must have

$$\frac{\beta^n}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\epsilon_n(w_{2k+2}^{\ell}) \cdot \epsilon_1(w_{2k+2}^{-\ell})}{\delta(\beta) - \delta(w_{2k+2}^{\ell})} = \frac{(1-\rho) \cdot \rho^n}{1 - \rho^{k+1}} \tag{112}$$

and this completes the proof. □

5.5 Mean Reflection Transience

Using the functional version of the Kolmogorov forward equations gives us the following expression for the mean transient queue length.

$$\frac{d}{dt}\mathrm{E}_m\left[Q_k(t)\right]$$
$$= \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} - \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\}$$
$$= \lambda \cdot \left(\frac{1-\rho^k}{1-\rho^{k+1}} - \frac{\beta^{k-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\sin\frac{k\ell\pi}{k+1} \cdot \left(\sin\frac{\ell m\pi}{k+1} - \beta \cdot \sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta \cdot \left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)} \cdot e^{-2\gamma\cdot(\delta(\beta) - \cos\frac{\ell\pi}{k+1})t} \right)$$
$$- \mu \cdot \left(\frac{\rho - \rho^{k+1}}{1-\rho^{k+1}} + \frac{\beta^{1-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\sin\frac{\ell\pi}{k+1} \cdot \left(\sin\frac{\ell m\pi}{k+1} - \beta \cdot \sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta \cdot \left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)} \cdot e^{-2\gamma\cdot(\delta(\beta) - \cos\frac{\ell\pi}{k+1})t} \right).$$

Now observe that in steady state, the flow into a queue equals the flow out or

$$\lambda \cdot \left(\frac{1-\rho^k}{1-\rho^{k+1}}\right) = \mu \cdot \frac{\rho - \rho^{k+1}}{1-\rho^{k+1}}. \tag{113}$$

Combining this with $\lambda = \gamma\beta$ and $\mu = \gamma/\beta$ gives us

$$\begin{aligned}
&\frac{d}{dt}\mathrm{E}_m\left[Q_k(t)\right] \\
&= \gamma\beta \cdot \frac{\beta^{k-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{(-1)^{\ell} \cdot \sin\frac{\ell\pi}{k+1} \cdot \left(\sin\frac{\ell m\pi}{k+1} - \beta \cdot \sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta \cdot \left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)} \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t} \\
&\quad - \frac{\gamma}{\beta} \cdot \frac{\beta^{1-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\sin\frac{\ell\pi}{k+1} \cdot \left(\sin\frac{\ell m\pi}{k+1} - \beta \cdot \sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta \cdot \left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)} \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t} \\
&= \frac{\gamma\beta^{-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\left(\beta^{k+1} \cdot (-1)^{\ell} - 1\right) \cdot \sin\frac{\ell\pi}{k+1} \cdot \left(\sin\frac{\ell m\pi}{k+1} - \beta \cdot \sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta \cdot \left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)} \cdot \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t}.
\end{aligned}$$

Integrating from t to infinity and taking the negative, we have

$$\begin{aligned}
\mathrm{E}_m[Q_k(t)] = &\ \frac{\rho}{1-\rho} - \frac{(k+1)\cdot\rho^{k+1}}{1-\rho^{k+1}} \\
&+ \frac{\beta^{-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\left(1 - \beta^{k+1} \cdot (-1)^{\ell}\right) \cdot \sin\frac{\ell\pi}{k+1} \cdot \left(\sin\frac{\ell m\pi}{k+1} - \beta \cdot \sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{2\beta \cdot \left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)^2} \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t},
\end{aligned} \tag{114}$$

where we now have the identity

$$\begin{aligned}
m = &\ \frac{\rho}{1-\rho} - \frac{(k+1)\cdot\rho^{k+1}}{1-\rho^{k+1}} \\
&+ \frac{\beta^{-m}}{k+1} \cdot \sum_{\ell=1}^{k} \frac{\left(1 - \beta^{k+1} \cdot (-1)^{\ell}\right) \cdot \sin\frac{\ell\pi}{k+1} \cdot \left(\sin\frac{\ell m\pi}{k+1} - \beta \cdot \sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{2\beta \cdot \left(\delta(\beta) - \cos\frac{\ell\pi}{k+1}\right)^2}.
\end{aligned} \tag{115}$$

5.6 Moments for Reflection Transience

Using the proof technique from computing the mean, we show how to use the same technique to derive expressions for any moment of the queue length. We first start with deriving expressions for the functional Kolmogorov forward equations below.

Lemma 5.5. *Let the time derivative of a function of the transient queue length have the following representation*

$$\frac{d}{dt}\mathrm{E}_m\left[f(Q_k(t))\right] = \sum_{j=1}^{k-1} a_j \cdot e^{-b_j t} \quad \text{where} \quad \lim_{t\to\infty} \mathrm{E}_m\left[f(Q_k(t))\right] = \mathrm{E}\left[f(Q_k)\right] \tag{116}$$

and $b_j > 0$ for all $j = 1, \ldots, k-1$. Then we have

$$\mathrm{E}_m\left[f(Q_k(t))\right] = f(m) + \sum_{j=1}^{k-1} \frac{a_j}{b_j} \cdot (1 - e^{-b_j t}) = \mathrm{E}\left[f(Q_k)\right] - \sum_{j=1}^{k-1} \frac{a_j}{b_j} \cdot e^{-b_j t}, \tag{117}$$

where

$$\sum_{j=1}^{k-1} \frac{a_j}{b_j} = \mathrm{E}\left[f(Q_k)\right] - f(m). \tag{118}$$

Proof. The proof follows immediately from integration of the time derivative from 0 to t. □

The above result is useful as it shows that a nice recursion can be developed for higher moments in terms of lower moments and the distribution of the queue length at 0 and k. Thus, we can use this to derive a new recursion for the scaled factorial moments of the $M/M/1/k$ queue. The following expression derives a recursion for the scaled factorial moments of the $M/M/1/k$ queue.

Theorem 5.6. *The time derivative of the l^{th} scaled factorial moment of the $M/M/1/k$ queue is given by*

$$\begin{aligned}\frac{d}{dt}\mathrm{E}_m\left[\binom{Q_k(t)}{\ell}\right] &= \lambda \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{\ell-1}\right] + \mu \cdot \sum_{j=0}^{\ell-1}(-1)^{\ell-j} \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{j}\right] \\ &\quad - \lambda \cdot \binom{k}{\ell-1} \cdot \mathrm{P}_m\{Q_k(t)=k\} + \mu \cdot (-1)^{\ell-1} \cdot \mathrm{P}_m\{Q_k(t)=0\}. \end{aligned} \tag{119}$$

Proof. Using the generalized binomial coefficient identities,

$$\binom{n}{\ell} = \binom{n-1}{\ell} + \binom{n-1}{\ell-1} \quad \text{and} \quad \binom{-1}{\ell} = (-1)^{\ell}, \tag{120}$$

gives us

$$\begin{aligned}\frac{d}{dt}\mathrm{E}_m\left[\binom{Q_k(t)}{\ell}\right] &= \lambda \cdot \mathrm{E}_m\left[\binom{Q_k(t)+1}{\ell} - \binom{Q_k(t)}{\ell}; Q_k(t) < k\right] \\ &\quad + \mu \cdot \mathrm{E}_m\left[\binom{Q_k(t)-1}{\ell} - \binom{Q_k(t)}{\ell}; Q_k(t) > 0\right] \\ &= \lambda \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{\ell-1}; Q_k(t) < k\right] - \mu \cdot \mathrm{E}_m\left[\binom{Q_k(t)-1}{\ell-1}; Q_k(t) > 0\right]\end{aligned}$$

and finally

$$\begin{aligned}\frac{d}{dt}\mathrm{E}_m\left[\binom{Q_k(t)}{\ell}\right] &= \lambda \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{\ell-1}\right] - \mu \cdot \mathrm{E}_m\left[\binom{Q_k(t)-1}{\ell-1}\right] \\ &\quad - \lambda \cdot \binom{k}{\ell-1} \cdot \mathrm{P}_m\{Q_k(t)=k\} + \mu \cdot (-1)^{\ell-1} \cdot \mathrm{P}_m\{Q_k(t)=0\} \\ &= \lambda \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{\ell-1}\right] + \mu \cdot \sum_{j=0}^{\ell-1}(-1)^{\ell-j} \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{j}\right] \\ &\quad - \lambda \cdot \binom{k}{\ell-1} \cdot \mathrm{P}_m\{Q_k(t)=k\} + \mu \cdot (-1)^{\ell-1} \cdot \mathrm{P}_m\{Q_k(t)=0\}.\end{aligned}$$

□

We can use these results to obtain a differential equation for the variance.

Corollary 5.7. *For all t, we have*

$$\begin{aligned}\frac{d}{dt}\mathrm{Var}_m\left[Q_k(t)\right] = & -2\left(\lambda \cdot \mathrm{E}_m\left[k - Q_k(t)\right] \cdot \mathrm{P}_m\left\{Q_k(t) = k\right\} + \mu \cdot \mathrm{E}_m\left[Q_k(t)\right] \cdot \mathrm{P}_m\left\{Q_k(t) = 0\right\}\right) \\ & + \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} + \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\}.\end{aligned}$$

Proof. For the cases of $\ell = 1$ and $\ell = 2$, we have

$$\frac{d}{dt}\mathrm{E}_m\left[Q_k(t)\right] = \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} - \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\} \tag{121}$$

and

$$\frac{d}{dt}\mathrm{E}_m\left[\binom{Q_k(t)}{2}\right] = \lambda \cdot \mathrm{E}_m\left[Q_k(t); Q_k(t) < k\right] - \mu \cdot \mathrm{E}_m\left[Q_k(t) - 1; Q_k(t) > 0\right] \tag{122}$$

This leads us to an equation for the variance

$$\begin{aligned}\frac{d}{dt}\mathrm{Var}_m\left[Q_k(t)\right] = & \, 2\lambda \cdot \mathrm{Cov}_m\left[Q_k(t), \{Q_k(t) < k\}\right] - 2\mu \cdot \mathrm{Cov}_m\left[Q_k(t), \{Q_k(t) > 0\}\right] \\ & + \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} + \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\},\end{aligned} \tag{123}$$

since

$$\begin{aligned}\frac{d}{dt}\mathrm{Var}_m[Q_k(t)] &= \frac{d}{dt}\mathrm{E}_m\left[2\binom{Q_k(t)}{2} + Q_k(t)\right] - 2\,\mathrm{E}_m\left[Q_k(t)\right] \cdot \frac{d}{dt}\mathrm{E}_m\left[Q_k(t)\right] \\ &= 2\lambda \cdot \mathrm{E}_m\left[Q_k(t); \{Q_k(t) < k\}\right] - 2\mu \cdot \mathrm{E}_m\left[Q_k(t) - 1; \{Q_k(t) > 0\}\right] \\ & \quad + \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} - \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\} \\ & \quad - 2 \cdot \mathrm{E}_m\left[Q_k(t)\right] \cdot \left(\lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} - \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\}\right).\end{aligned}$$

We can then combine the first and third lines as covariances to obtain

$$\begin{aligned}\frac{d}{dt}\mathrm{Var}_m\left[Q_k(t)\right] &= 2\lambda \cdot \mathrm{Cov}_m\left[Q_k(t), \{Q_k(t) < k\}\right] - 2\mu \cdot \mathrm{Cov}_m\left[Q_k(t), \{Q_k(t) > 0\}\right] \\ & \quad + \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} + \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\} \\ &= 2\lambda \cdot \left(\mathrm{E}_m\left[Q_k(t)\right] - k\right) \cdot \mathrm{P}_m\left\{Q_k(t) = k\right\} - 2\mu \cdot \mathrm{E}_m\left[Q_k(t)\right] \cdot \mathrm{P}_m\left\{Q_k(t) = 0\right\} \\ & \quad + \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} + \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\}.\end{aligned}$$

□

The transient recursion for the moments relies on knowing the transient behavior of the queue length distribution at the empty state $\{0\}$ and the full state $\{k\}$. Unlike in the infinite server case, the $M/M/1/k$ moments are not a closed system of differential equations.

6 Summary of Results

In this section, we provide a summary of the results obtained in the paper. This provides a thorough collection of results and provides a complete list of formulas for the transient analysis of the $M/M/1/k$ queue.

6.1 Free Transience

For all integers m and n, we have

$$\mathrm{P}_m \{Z(t) = n\} = \frac{\beta^{n-m}}{\pi} \cdot \int_0^\pi \cos((m-n)\cdot\theta) \cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\theta)t}\, d\theta. \tag{124}$$

6.2 Absorbing Transience for the Transient States

For all states $0 < m < k$ and $0 < n < k$, we have

$$\mathrm{P}_m \{Q_k(t) = n, T_{0.k} > t\} = \frac{2\beta^{n-m}}{k} \cdot \sum_{\ell=1}^{k-1} \sin\frac{\ell m\pi}{k} \cdot \sin\frac{\ell n\pi}{k} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi\ell}{k}\right)t}. \tag{125}$$

After summing over all the transient states of $0 < n < k$, we have

$$\mathrm{P}_m \{T_{0,k} > t\} = \frac{\beta^{-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \left(\sin\frac{\ell\pi}{k} + \beta^k \cdot \sin\frac{(k-1)\ell\pi}{k}\right)}{\delta(\beta) - \cos\frac{\ell\pi}{k}} \cdot e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t} \tag{126}$$

and

$$\mathrm{E}_m [T_{0,k} \wedge t] = \frac{\beta^{-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \left(\sin\frac{\ell\pi}{k} + \beta^k \cdot \sin\frac{(k-1)\ell\pi}{k}\right)}{\left(\delta(\beta) - \cos\frac{\ell\pi}{k}\right)^2} \cdot \left(1 - e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t}\right). \tag{127}$$

This also gives us a formula for the mean of the absorbing process since

$$\mathrm{E}_m [Q_k^*(t)] = m + (\lambda - \mu) \cdot \mathrm{E}_m [T_{0,k} \wedge t]. \tag{128}$$

6.3 Absorbing Transience for the Absorbing States

For all integers $0 < m < k$, we have

$$\mathrm{P}_m \{Q_k(T_{0,k}) = 0, T_{0.k} \le t\} = \frac{\beta^{-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \sin\frac{\ell\pi}{k}}{\delta(\beta) - \cos\frac{\ell\pi}{k}} \cdot \left(1 - e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t}\right). \tag{129}$$

and

$$\mathrm{P}_m \{Q_k(T_{0,k}) = k, T_{0.k} \le t\} = \frac{\beta^{k-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \sin\frac{(k-1)\ell\pi}{k}}{\delta(\beta) - \cos\frac{\ell\pi}{k}} \cdot \left(1 - e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k}\right)t}\right). \tag{130}$$

This gives us

$$\mathrm{P}_m \{Q_k(T_{0,k}) = 0\} = \frac{\beta^{-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \sin\frac{\ell\pi}{k}}{\delta(\beta) - \cos\frac{\ell\pi}{k}}. \tag{131}$$

and

$$\mathrm{P}_m \{Q_k(T_{0,k}) = k\} = \frac{\beta^{k-m}}{k} \cdot \sum_{\ell=1}^{k-1} \frac{\sin\frac{\ell m\pi}{k} \cdot \sin\frac{(k-1)\ell\pi}{k}}{\delta(\beta) - \cos\frac{\ell\pi}{k}}. \tag{132}$$

6.4 Quasi-Steady State Absorbing Transience

The quasi-steady state distribution for the absorbing process is

$$\lim_{t\to\infty} \mathrm{P}_m \left\{Q_k(t) = n \,|T_{0,k} > t\right\} = \frac{2\beta^n \cdot \left(\delta(\beta) - \cos\frac{\pi}{k}\right) \cdot \sin\frac{n\pi}{k}}{(1+\beta^k)\cdot \sin\frac{\pi}{k}}. \tag{133}$$

Initializing with this distribution gives us

$$\mathrm{P}_* \left\{Q_k(t) = n \,|T_{0,k} > t\right\} = \frac{2\beta^n \cdot \left(\delta(\beta) - \cos\frac{\pi}{k}\right) \cdot \sin\frac{n\pi}{k}}{(1+\beta^k)\cdot \sin\frac{\pi}{k}}, \tag{134}$$

for all $t \geq 0$. We also have

$$\mathrm{P}_* \left\{T_{0,k} > t\right\} = e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi}{k}\right)t} \quad \text{and} \quad \mathrm{E}_*\left[T_{0,k}\wedge t\right] = \frac{1 - e^{-2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi}{k}\right)t}}{2\gamma\cdot\left(\delta(\beta)-\cos\frac{\pi}{k}\right)}. \tag{135}$$

Combining these results also gives us the following mean behavior

$$\mathrm{E}_*\left[Q_k\left(T_{0,k}\wedge t\right)\right] = \frac{k\beta^k}{1+\beta^k} - \frac{2\,\epsilon_1(\beta)\cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\pi}{k})t}}{\delta(\beta)-\cos\frac{\pi}{k}}. \tag{136}$$

This formula leads to two special cases:

$$\mathrm{E}_*\left[Q_k\left(0\right)\right] = \frac{k\beta^k}{1+\beta^k} - \frac{2\,\epsilon_1(\beta)}{\delta(\beta)-\cos\frac{\pi}{k}} \quad \text{and} \quad \mathrm{E}_*\left[Q_k\left(T_{0,k}\right)\right] = \frac{k\beta^k}{1+\beta^k}. \tag{137}$$

6.5 Reflection Transience

For all integers $0 \leq m \leq k$ and $0 \leq n \leq k$, we have

$$\begin{aligned}\mathrm{P}_m\{Q_k(t) = n\} &= \frac{(1-\rho)\cdot\rho^n}{1-\rho^{k+1}} \\ &+ \frac{\beta^{n-m}}{k+1}\cdot\sum_{\ell=1}^{k} \frac{\left(\sin\frac{\ell n\pi}{k+1} - \beta\cdot\sin\frac{\ell\cdot(n+1)\pi}{k+1}\right)\cdot\left(\sin\frac{\ell m\pi}{k+1} - \beta\cdot\sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{\beta\cdot\left(\delta(\beta)-\cos\frac{\ell\pi}{k+1}\right)}\cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t}\end{aligned} \tag{138}$$

and

$$\begin{aligned}\mathrm{E}_m\left[Q_k(t)\right] &= \frac{\rho}{1-\rho} - \frac{(k+1)\cdot\rho^{k+1}}{1-\rho^{k+1}} \\ &+ \frac{\beta^{-m}}{k+1}\cdot\sum_{\ell=1}^{k}\frac{\left(1-\beta^{k+1}\cdot(-1)^\ell\right)\cdot\sin\frac{\ell\pi}{k+1}\cdot\left(\sin\frac{\ell m\pi}{k+1} - \beta\cdot\sin\frac{\ell\cdot(m+1)\pi}{k+1}\right)}{2\beta\cdot\left(\delta(\beta)-\cos\frac{\pi\ell}{k+1}\right)^2}\cdot e^{-2\gamma\cdot(\delta(\beta)-\cos\frac{\ell\pi}{k+1})t}.\end{aligned} \tag{139}$$

For binomial coefficient moments of the reflection process, we have the following set of differential equations

$$\frac{d}{dt}\mathrm{E}_m\left[\binom{Q_k(t)}{\ell}\right] = \lambda \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{\ell-1}\right] + \mu \cdot \sum_{j=0}^{\ell-1}(-1)^{\ell-j} \cdot \mathrm{E}_m\left[\binom{Q_k(t)}{j}\right] \tag{140}$$
$$- \lambda \cdot \binom{k}{\ell-1} \cdot \mathrm{P}_m\{Q_k(t) = k\} + \mu \cdot (-1)^{\ell-1} \cdot \mathrm{P}_m\{Q_k(t) = 0\}.$$

As special cases, we then have

$$\frac{d}{dt}\mathrm{E}_m\left[Q_k(t)\right] = \lambda \cdot \mathrm{P}_m\left\{Q_k(t) < k\right\} - \mu \cdot \mathrm{P}_m\left\{Q_k(t) > 0\right\} \tag{141}$$

and

$$\frac{d}{dt}\mathrm{Var}_m[Q_k(t)] = -2\left(\lambda \cdot \mathrm{E}_m\left[k - Q_k(t)\right] \cdot \mathrm{P}_m\{Q_k(t) = k\} + \mu \cdot \mathrm{E}_m\left[Q_k(t)\right] \cdot \mathrm{P}_m\{Q_k(t) = 0\}\right)$$
$$+ \lambda \cdot \mathrm{P}_m\{Q_k(t) < k\} + \mu \cdot \mathrm{P}_m\{Q_k(t) > 0\}\,. \tag{142}$$

7 Conclusion

This paper develops a unified framework of techniques for deriving a complete transient analysis of the $M/M/1/k$ queue. Our methodology is based on the Markovian behavior of the queuing process and augmented by group theoretic and complex analytic techniques. Many of our new performance measures are inspired by bike sharing services and systems. The results have applications to other types of resource-sharing services.

A natural area of interest is to extend our results to networks of $M/M/1/k$ queues to apply them to a bike sharing systems with large numbers of stations. Our analysis can extend to those systems, by using multi-dimensional complex integrals. The absorbing process in this multi-dimensional case now reflects the time to rebalance over this larger system. It is also of interest to extend our work to queues where the arrivals could be self-exciting or time varying. In this context, the work of [11, 12, 21] would be helpful in defining and analyzing such systems.

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References

[1] Eitan Altman and Alain Jean-Marie. The loss process of messages in an $M/M/1/k$ queue. In *Proceedings of INFOCOM'94 Conference on Computer Communications*, pages 1191–1198. IEEE, 1994.

[2] F. Baccelli and W. Massey. On the busy period of certain classes of queueing networks. In *Computer Performance and Reliability*, 1987.

[3] François Baccelli, William A. Massey, and Paul E. Wright. Determining the exit time distribution for a closed cyclic network. *Theor. Comput. Sci.*, 125(1):149–165, 1994. doi: 10.1016/0304-3975(94)90298-4. URL https://doi.org/10.1016/0304-3975(94)90298-4.

[4] Elisabetta Biondi, Chiara Boldrini, and Raffaele Bruno. The impact of regulated electric fleets on the power grid: The car sharing case. In *2016 IEEE 2nd International Forum on Research and Technologies for Society and Industry Leveraging a better tomorrow (RTSI)*, pages 1–6. IEEE, 2016.

[5] Giuseppe C Calafiore, Francesco Portigliotti, and Alessandro Rizzo. A network model for an urban bike sharing system. *IFAC-PapersOnLine*, 50(1):15633–15638, 2017.

[6] Daniel Chemla, Frédéric Meunier, and Roberto Wolfler Calvo. Bike sharing systems: Solving the static rebalancing problem. *Discrete Optimization*, 10(2):120–146, 2013.

[7] Mengjing Chen, Weiran Shen, Pingzhong Tang, and Song Zuo. Optimal vehicle dispatching for ride-sharing platforms via dynamic pricing. In *Companion Proceedings of the The Web Conference 2018*, pages 51–52, 2018.

[8] Hangil Chung, Daniel Freund, and David B Shmoys. Bike angels: An analysis of citi bike's incentive program. In *Proceedings of the 1st ACM SIGCAS Conference on Computing and Sustainable Societies*, pages 1–9, 2018.

[9] Jacob Willem Cohen. *The single server queue*. Elsevier, 2012.

[10] JW Cohen. On the busy periods for the M/G/1 queue with finite and with infinite waiting room. *Journal of Applied Probability*, pages 821–827, 1971.

[11] Andrew Daw and Jamol Pender. An ephemerally self-exciting point process. *arXiv preprint arXiv:1811.04282*, 2018.

[12] Andrew Daw and Jamol Pender. Queues driven by hawkes processes. *Stochastic Systems*, 8(3):192–229, 2018.

[13] Ahmadreza Faghih-Imani, Robert Hampshire, Lavanya Marla, and Naveen Eluru. An empirical analysis of bike sharing usage and rebalancing: Evidence from barcelona and seville. *Transportation Research Part A: Policy and Practice*, 97:177–191, 2017.

[14] Willliam Feller. *An introduction to probability theory and its applications, vol 2.* John Wiley & Sons, 2008.

[15] Christine Fricker and Nicolas Gast. Incentives and redistribution in homogeneous bike-sharing systems with stations of finite capacity. *Euro journal on transportation and logistics*, 5(3):261–291, 2016.

[16] Christine Fricker, Nicolas Gast, and Hanene Mohamed. Mean field analysis for inhomogeneous bike sharing systems. In *Discrete Mathematics and Theoretical Computer Science*, 2012.

[17] Frank N Gouweleeuw. The loss probability in an overloaded queue using the dual queue. *Operations research letters*, 21(2):101–106, 1997.

[18] Robert C Hampshire and Lavanya Marla. An analysis of bike sharing usage: Explaining trip generation and attraction from observed demand. In *91st Annual meeting of the transportation research board, Washington, DC*, pages 12–2099, 2012.

[19] Sophie Hautphenne and Moshe Haviv. The Bias Optimal k in the $M/M/1/k$ Queue: An Application of the Deviation Matrix. *Probability in the Engineering and Informational Sciences*, 30(1):61, 2016.

[20] Jianbin Huang, Heli Sun, He Li, Longji Huang, Ao Li, and Xiangyu Wang. Central station-based demand prediction for determining target inventory in a bike-sharing system. *The Computer Journal*, 2020.

[21] David T Koops, Mayank Saxena, Onno J Boxma, and Michel Mandjes. Infinite-server queues with hawkes input. *Journal of Applied Probability*, 55(3):920–943, 2018.

[22] Katarzyna Kosek-Szott. Throughput, delay, and frame loss probability analysis of IEEE 802.11 DCF with $M/M/1/k$ queues. In *2013 IEEE 24th Annual International Symposium on Personal, Indoor, and Mobile Radio Communications (PIMRC)*, pages 2234–2238. IEEE, 2013.

[23] Benjamin Legros. Dynamic repositioning strategy in a bike-sharing system; how to prioritize and how to rebalance a bike station. *European Journal of Operational Research*, 272(2):740–753, 2019.

[24] Quan-Lin Li and Rui-Na Fan. A mean-field matrix-analytic method for bike sharing systems under markovian environment. *Annals of Operations Research*, pages 1–35, 2021.

[25] Quan-Lin Li, Chang Chen, Rui-Na Fan, Liang Xu, and Jing-Yu Ma. Queueing analysis of a large-scale bike sharing system through mean-field theory. *arXiv preprint arXiv:1603.09560*, 2016.

[26] Quan-Lin Li, Rui-Na Fan, and Jing-Yu Ma. A unified framework for analyzing closed queueing networks in bike sharing systems. In *International conference on information technologies and mathematical modelling*, pages 177–191. Springer, 2016.

[27] Quan-Lin Li, Rui-Na Fan, and Zhi-Yong Qian. A nonlinear solution to closed queueing networks for bike sharing systems with markovian arrival processes and under an irreducible path graph. In *International conference on queueing theory and network applications*, pages 118–140. Springer, 2017.

[28] Quan-Lin Li, Zhi-Yong Qian, and Rui-Na Fan. Fluid and diffusion limits for bike sharing systems. In *International conference on queueing theory and network applications*, pages 217–245. Springer, 2017.

[29] William A Massey. Calculating exit times for series jackson networks. *Journal of applied probability*, pages 226–234, 1987.

[30] William A Massey. Stochastic orderings for markov processes on partially ordered spaces. *Mathematics of operations research*, 12(2):350–367, 1987.

[31] Jolly Mathers. The barber queue. *The Mathematics Teacher*, 69(8):680–684, 1976.

[32] Rahul Nair, Elise Miller-Hooks, Robert C Hampshire, and Ana Bušić. Large-scale vehicle sharing systems: analysis of Vélib'. *International Journal of Sustainable Transportation*, 7(1):85–106, 2013.

[33] Eoin O'Mahony and David Shmoys. Data analysis and optimization for (Citi) bike sharing. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 29, 2015.

[34] Ling Pan, Qingpeng Cai, Zhixuan Fang, Pingzhong Tang, and Longbo Huang. A deep reinforcement learning framework for rebalancing dockless bike sharing systems. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 1393–1400, 2019.

[35] Jamol Pender. Nonstationary loss queues via cumulant moment approximations. *Probability in the Engineering and Informational Sciences*, 29(1):27, 2015.

[36] Jamol Pender, Shuang Tao, and Anders Wikum. A stochastic model for electric scooter systems. *Available at SSRN 3582320*, 2020.

[37] Tal Raviv and Ofer Kolka. Optimal inventory management of a bike-sharing station. *Iie Transactions*, 45(10):1077–1093, 2013.

[38] Rhonda Righter. A note on losses in M/GI/1/n queues. *Journal of applied probability*, pages 1240–1243, 1999.

[39] Stig I Rosenlund. On the length and number of served customers of the busy period of a generalised m/g/1 queue with finite waiting room. *Advances in Applied Probability*, pages 379–389, 1973.

[40] Bacem Samet, Florent Couffin, Maher Barkallah, Marc Zolghadri, and Mohamed Haddar. Performance evaluation using closed queueing network in bike sharing systems. In *International Conference on Acoustics and Vibration, ICAV2016*, 2016.

[41] Bacem Samet, Florent Couffin, Marc Zolghadri, Maher Barkallah, and Mohamed Haddar. Performance analysis and improvement of the bike sharing system using closed queuing networks with blocking mechanism. *Sustainability*, 10(12):4663, 2018.

[42] Jasper Schuijbroek, Robert C Hampshire, and W-J Van Hoeve. Inventory rebalancing and vehicle routing in bike sharing systems. *European Journal of Operational Research*, 257(3):992–1004, 2017.

[43] JG Skellam. The frequency distribution of the difference between two poisson variates belonging to different populations. *Journal of the Royal Statistical Society. Series A (General)*, 109(Pt 3):296–296, 1946.

[44] Lajos Takács. *Introduction to the theory of queues*. University Texts in the Mathematical Sciences. Oxford University Press, New York, 1962.

[45] Shuang Tao. *Limit Theorems in Queueing Networks with Applications to Shared Mobility and Healthcare*. PhD thesis, Cornell University, 2020.

[46] Shuang Tao and Jamol Pender. The impact of smartphone apps on bike sharing systems. *Available at SSRN 3582275*, 2020.

[47] Shuang Tao and Jamol Pender. A stochastic analysis of bike-sharing systems. *Probability in the Engineering and Informational Sciences*, page 1–58, 2020. doi: 10.1017/S0269964820000297.

[48] Richard A Upton and Satish K Tripathi. An approximate transient analysis of the M(t)/M/1 queue. *Performance Evaluation*, 2(2):118–132, 1982.

[49] Patrick Vogel, Torsten Greiser, and Dirk Christian Mattfeld. Understanding bike-sharing systems using data mining: Exploring activity patterns. *Procedia-Social and Behavioral Sciences*, 20:514–523, 2011.

[50] Ariel Waserhole, Vincent Jost, and Nadia Brauner. Pricing techniques for self regulation in vehicle sharing systems. *Electronic Notes in Discrete Mathematics*, 41:149–156, 2013.