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The truncated normal distribution: Applications to queues with impatient customers

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ABSTRACT

Motivated by heavy traffic approximations for single server queues with abandonment, we provide an exact expression for the moments of the truncated normal distribution using Stein's lemma. Consequently, our moment expressions provide insight into the steady state skewness and kurtosis dynamics of single server queues with impatient customers. Moreover, based on the truncated normal distribution, we develop a new approximation for single server queues with abandonment in the nonstationary setting. Numerical examples illustrate that our approximation performs quite well.

1. Introduction

The truncated normal distribution is a very important distribution in the world of probability and statistics. It appears quite naturally when the normal distribution itself arises. For example, when one wants to *threshold* or *screen* values from a dataset that is normally distributed, the remaining data has a truncated normal distribution. Therefore to analyze the moments of the remaining data, one needs to study the moments of the truncated normal distribution.

Most if not all of the available literature tends to focus on the mean and variance of the truncated normal distribution. This is partially motivated from the statistical community since they are interested in obtaining unbiased mean and variance estimators for data that is screened or thresholded. See for example [1–3,5]. In this paper, we not only provide exact expressions for the skewness and kurtosis, but also provide any moment of the truncated normal distribution. Later in the paper, we also use the truncated normal distribution to approximate the nonstationary single server queue with abandonment.

Although there is substantial motivation to study the moments of the truncated normal distribution from a statistical perspective, we are primarily motivated by developing approximations for the cumulant moments of queues with impatient customers. There is a large and growing literature on queues with impatient customers, for instance, [21,22] show that the truncated normal distribution arises as the heavy traffic diffusion limit for the stationary single server queue with impatient customers. More recently, [6] showed that the truncated normal distribution is the heavy traffic limit of ticket queues where customers are unobservable. In [22] they consider a GI/GI/1 + GI queueing model with abandonment. They assume that the server works at rate one under the FIFO discipline. The primitives of the model include three independent sequences of non-negative i.i.d. random variables for the inter arrival times, service times, and abandonment times. We assume that the service times have mean $\frac{1}{\mu}$ and coefficient of variation σ_s. The inter arrival times have mean $\frac{1}{\lambda} = \frac{1}{\mu+\beta\cdot\sqrt{\mu}}$ and coefficient of variation σ_a where β is the heavy traffic parameter. Lastly, we assume that the abandonment can have any distribution where the derivative of cdf evaluated at zero is strictly positive with value θ. The main theorem proved in [22] says the following:

Theorem 1.1 (*[22]*). *If*

$$\tilde{Q}^n(0) \Rightarrow \tilde{Q}_0, \quad \text{as } n \to \infty,$$

then we have the following convergence for the queue length process and generalized linear regulator mapping $(\tilde{Q}^n, \tilde{Y}^n)$ *as described in Eq.* 3.3 *in* [22]

$$(\tilde{Q}^n, \tilde{Y}^n) \Rightarrow (\tilde{Q}, \tilde{Y}), \quad \text{as } n \to \infty,$$

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where $\tilde{Q}(0)$ is equal in distribution to $\tilde{Q}_0$ and together $\tilde{Q}$ and $\tilde{Y}$ obey the following stochastic differential equation:

$$d\tilde{Q}(t) = (-\beta + \theta \cdot \tilde{Q}(t))dt + \sigma \cdot dB(t) + d\tilde{Y}(t)$$

where β is the scaled heavy traffic scaling parameter, θ is the derivative of the abandonment distribution at zero, and $\sigma^2 = \sigma_a^2 + \sigma_s^2$ is the sum of the arrival and service distributions coefficient of variation.

The process $\tilde{Q}(t)$ is known as a regulated Ornstein–Uhlenbeck (ROU) process and the steady state distribution of $\tilde{Q}(t)$ is a truncated normal random variable that is conditioned or regulated to be in the interval $(0, \infty)$

$$\tilde{Q}(\infty) = \text{Normal}\left(\frac{\beta}{\theta}, \frac{\sigma^2}{2\theta}, 0, \infty\right). \tag{1.1}$$

In the work of [6,22], they only analyze the steady state mean dynamics. However, it is important to analyze higher cumulants such as the variance, skewness, and kurtosis as they provide essential insights into the behavior of the queueing process. Since the Gaussian is defined to have zero skewness and zero excess kurtosis, when the skewness and kurtosis are far from zero, it implies that a Gaussian approximation of the dynamics might not be appropriate. In fact, the work of [9–11,15–19] shows that the skewness and kurtosis can play a significant role in estimating queueing performance. Thus, we believe that our exact expressions for the higher cumulants of the truncated normal will give us insight into the dynamics of queues with impatient customers.

2. Stein's lemma and main results

2.1. Stein's lemma

In this section, we give a brief overview of Hermite polynomials and Stein's lemma [20], which are important ingredients for deriving our exact expressions for the moments of the truncated normal distribution. The probabilistic Hermite polynomials as described in [14] are defined as:

$$h_n(x) = \frac{1}{\varphi(x)} \cdot \left(-\frac{d}{dx}\right)^n \varphi(x).$$

The first four Hermite polynomials are

$$h_0(x) = 1, \qquad h_1(x) = x, \qquad h_2(x) = x^2 - 1,$$
$$h_3(x) = x^3 - 3x,$$

and in general they solve the recurrence relation

$$h_{n+1}(x) = x \cdot h_n(x) - n \cdot h_{n-1}(x).$$

We have the following Hermite polynomial generalization of Stein's lemma; however, for the remainder of the paper, the random variable X is a standard Gaussian random variable.

Lemma 2.1. *If X is a standard Gaussian random variable and $E[f^{(n)}(X)] < \infty$, then*

$$E\left[f(X) \cdot h_n(X)\right] = E[f^{(n)}(X)]$$

where f is any generalized function and $f^{(n)}$ is the nth derivative of the function f.

For example, since $\{X \geq \chi\}$ is a generalized function, Stein's lemma can be used to obtain

$$E\left[X \cdot \{X \geq \chi\}\right] = E\left[\delta_\chi(X)\right] = \varphi(\chi),$$

or for $n \geq 1$

$$E\left[h_n(X) \cdot \{X \geq \chi\}\right] = E\left[h_{n-1}(X) \cdot \delta_\chi(X)\right] = h_{n-1}(\chi) \cdot \varphi(\chi),$$

where we define φ and Φ to be the density and the cumulative distribution functions, respectively, for $X \sim \text{Normal}(0, 1)$, i.e.,

$$\varphi(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \Phi(x) \equiv \int_{-\infty}^{x} \varphi(y)\, dy,$$

and let $\overline{\Phi}(x) \equiv \int_x^\infty \varphi(y)\, dy$.

In addition to the derivative properties of the Hermite polynomials, it is well known from [4] that the probabilistic Hermite polynomials have the following explicit form in terms of standard polynomials,

$$h_n(x) = n! \cdot \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m! \cdot (n-2m)!} \cdot \frac{x^{n-2m}}{2^m}. \tag{2.2}$$

However, the above relation of Eq. (2.2) can be inverted to represent any polynomial in terms of the Hermite polynomials as the next theorem shows.

Theorem 2.2. *Any polynomial has the following decomposition representation in terms of the probabilist Hermite polynomials*

$$X^n = n! \cdot \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{2^{-m}}{m! \cdot (n-2m)!} \cdot h_{n-2m}(X).$$

Proof. This proof follows from induction and exploiting the Rodrigues recursion relation for Hermite polynomials. We also provide a proof of this result in an online appendix to this paper found on the author's website. □

This representation of any polynomial in terms of a sum of Hermite polynomials will be useful since any moment of a Gaussian random variable can be computed by the last coefficient of the sum since all Hermite polynomials where $n \geq 1$ have expectation zero. Now with our review of Hermite polynomials and their properties, we give our main result.

Theorem 2.3. *Suppose that Q has a normal distribution with mean q and variance v; then the nth conditional moment has representation*

$$E[Q^n | a \leq Q \leq b] = \frac{\sum_{j=0}^{n} a_j \cdot \left(\sum_{m=0}^{\lfloor j/2 \rfloor} b_{jm} \cdot (h_{n-2m-1}(\chi) \cdot \varphi(\chi) - h_{n-2m-1}(\psi) \cdot \varphi(\psi))\right)}{\Phi(\psi) - \Phi(\chi)}$$

where we define

$$a_j = \binom{n}{j} \cdot \sqrt{v^j} \cdot q^{n-j} \cdot j! \quad \text{and} \quad b_{jm} = \frac{2^{-m}}{m! \cdot (j-2m)!},$$
$$(h_{-1}(\chi) \cdot \varphi(\chi) - h_{-1}(\psi) \cdot \varphi(\psi)) = \Phi(\psi) - \Phi(\chi),$$

and

$$\chi = \frac{a-q}{\sqrt{v}} \quad \text{and} \quad \psi = \frac{b-q}{\sqrt{v}}.$$

Proof. We prove this result in an online appendix to this paper found on the author's website. □

As a result of the above expression, we have the following corollary, which gives explicit expressions for the mean, variance, skewness, and kurtosis of the truncated normal distribution.

Corollary 2.4. *Eqs. (2.3), (2.4),* Skew$[Q | a \leq Q \leq b]$ *and* Kurt$[Q | a \leq Q \leq b]$ *are given in Box I.*

Proof. After some tedious calculations which we omit for brevity, the proof follows from using Theorem 2.3 and understanding the definitions of the variance, skewness, and kurtosis of random variables. □

$$E[Q|a \le Q \le b] = q + \sqrt{v} \cdot \frac{\varphi(\chi) - \varphi(\psi)}{\Phi(\psi) - \Phi(\chi)} \tag{2.3}$$

$$\mathrm{Var}[Q|a \le Q \le b] = v + v \cdot \frac{\chi \cdot \varphi(\chi) - \psi \cdot \varphi(\psi)}{\Phi(\psi) - \Phi(\chi)} - v \cdot \frac{(\varphi(\chi) - \varphi(\psi))^2}{(\Phi(\psi) - \Phi(\chi))^2} \tag{2.4}$$

$$\mathrm{Skew}[Q|a \le Q \le b] = \frac{\left(\frac{(h_2(\chi)\cdot\varphi(\chi) - h_2(\psi)\cdot\varphi(\psi))}{\Phi(\psi)-\Phi(\chi)} - 3 \cdot \frac{(\chi\cdot\varphi(\chi)-\psi\cdot\varphi(\psi))\cdot(\varphi(\chi)-\varphi(\psi))}{(\Phi(\psi)-\Phi(\chi))^2} + 2 \cdot \frac{(\varphi(\chi)-\varphi(\psi))^3}{(\Phi(\psi)-\Phi(\chi))^3} \right)}{\left(1 + \frac{\chi\cdot\varphi(\chi)-\psi\cdot\varphi(\psi)}{\Phi(\psi)-\Phi(\chi)} - \frac{(\varphi(\chi)-\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^2}\right)^{3/2}}$$

$$\begin{aligned}\mathrm{Kurt}[Q|a \le Q \le b] \;=\; & \frac{\left(12 \cdot \frac{(h_1(\chi)\cdot\varphi(\chi)-h_1(\psi)\cdot\varphi(\psi))\cdot(\varphi(\chi)-\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^3}\right)}{\left(1 + \frac{\chi\cdot\varphi(\chi)-\psi\cdot\varphi(\psi)}{\Phi(\psi)-\Phi(\chi)} - \frac{(\varphi(\chi)-\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^2}\right)^2} \\ & - \frac{\left(4 \cdot \frac{(h_2(\chi)\cdot\varphi(\chi)-h_2(\psi)\cdot\varphi(\psi))\cdot(\varphi(\chi)-\varphi(\psi))}{(\Phi(\psi)-\Phi(\chi))^2}\right)}{\left(1 + \frac{\chi\cdot\varphi(\chi)-\psi\cdot\varphi(\psi)}{\Phi(\psi)-\Phi(\chi)} - \frac{(\varphi(\chi)-\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^2}\right)^2} - \frac{\left(3 \cdot \frac{(h_1(\chi)\cdot\varphi(\chi)-h_1(\psi)\cdot\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^2}\right)}{\left(1 + \frac{\chi\cdot\varphi(\chi)-\psi\cdot\varphi(\psi)}{\Phi(\psi)-\Phi(\chi)} - \frac{(\varphi(\chi)-\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^2}\right)^2} \\ & - \frac{\left(6 \cdot \frac{(\varphi(\chi)-\varphi(\psi))^4}{(\Phi(\psi)-\Phi(\chi))^4}\right)}{\left(1 + \frac{\chi\cdot\varphi(\chi)-\psi\cdot\varphi(\psi)}{\Phi(\psi)-\Phi(\chi)} - \frac{(\varphi(\chi)-\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^2}\right)^2} + \frac{\left(\frac{(h_3(\chi)\cdot\varphi(\chi)-h_3(\psi)\cdot\varphi(\psi))}{(\Phi(\psi)-\Phi(\chi))}\right)}{\left(1 + \frac{\chi\cdot\varphi(\chi)-\psi\cdot\varphi(\psi)}{\Phi(\psi)-\Phi(\chi)} - \frac{(\varphi(\chi)-\varphi(\psi))^2}{(\Phi(\psi)-\Phi(\chi))^2}\right)^2}.\end{aligned}$$

Box I.

Table 1
Truncated normal approximations (100 simulations).

Samples	a	b	q	v	Mean	Sim	Var	Sim	Skew	Sim	Kurt	Sim
10^4	0	∞	1	2	1.578	1.577 ± 0.0016	1.088	1.089 ± 0.0028	0.714	0.716 ± 0.0039	0.201	0.209 ± 0.0142
10^5	0	∞	1	2	1.578	1.577 ± 0.0010	1.088	1.0883 ± 0.0008	0.714	0.713 ± 0.0012	0.201	0.198 ± 0.0044
10^6	0	∞	1	2	1.578	1.578 ± 0.0002	1.088	1.0881 ± 0.0003	0.714	0.715 ± 0.0004	0.201	0.200 ± 0.0014
10^7	0	∞	1	2	1.578	1.578 ± 0.0001	1.088	1.0880 ± 0.0001	0.714	0.714 ± 0.0001	0.201	0.201 ± 0.0005
10^4	0	∞	10	20	10.148	10.153 ± 0.0059	18.494	18.486 ± 0.0356	0.157	0.156 ± 0.0028	−0.214	-0.211 ± 0.0057
10^5	0	∞	10	20	10.148	10.146 ± 0.0021	18.494	18.497 ± 0.0120	0.157	0.158 ± 0.0010	−0.214	-0.213 ± 0.0019
10^6	0	∞	10	20	10.148	10.146 ± 0.0007	18.494	18.510 ± 0.0045	0.157	0.157 ± 0.0003	−0.214	-0.215 ± 0.0007
10^7	0	∞	10	20	10.148	10.148 ± 0.0002	18.494	18.484 ± 0.0015	0.157	0.157 ± 0.0001	−0.214	-0.214 ± 0.0002
10^4	0	∞	100	200	100	99.991 ± 0.0180	200	199.682 ± 0.3814	0	0.002 ± 0.0036	0	-0.0029 ± 0.0069
10^5	0	∞	100	200	100	100.002 ± 0.0062	200	199.91 ± 0.1271	0	0.0004 ± 0.0010	0	0.0009 ± 0.0024
10^6	0	∞	100	200	100	99.987 ± 0.0021	200	198.983 ± 0.0425	0	-0.0002 ± 0.0003	0	-0.001 ± 0.0008
10^7	0	∞	100	200	100	99.993 ± 0.0008	200	200.101 ± 0.0147	0	0.0001 ± 0.0001	0	0.000 ± 0.0003

Table 2
GI/GI/1 + GI queue approximations (100 simulations).

Time	λ	θ	β	σ	Mean	Sim	Var	Sim	Skew	Sim	Kurt	Sim
10^4	100	0.1	1	$\sqrt{2}$	100.08	97.10 ± 0.25	991.48	991.55 ± 8.06	0.0246	0.0493 ± 0.011	−0.061	-0.068 ± 0.017
10^4	100	0.1	0	$\sqrt{2}$	25.23	25.07 ± 0.113	363.38	367.23 ± 3.371	0.995	0.992 ± 0.012	0.8692	0.849 ± 0.051
10^4	100	0.1	−1	$\sqrt{2}$	8.603	7.86 ± 0.023	65.69	64.90 ± 0.519	1.6968	1.768 ± 0.021	3.907	4.311 ± 0.145
10^4	100	0.2	1	$\sqrt{2}$	50.74	48.91 ± 0.12	462.36	459.88 ± 2.24	0.1576	0.193 ± 0.01	−0.2145	-0.197 ± 0.015
10^4	100	0.2	0	$\sqrt{2}$	17.84	17.58 ± 0.057	181.69	189.02 ± 1.43	0.9953	1.002 ± 0.001	0.8692	0.876 ± 0.0396
10^4	100	0.2	−1	$\sqrt{2}$	7.777	7.09 ± 0.017	50.65	51.07 ± 0.332	1.576	1.675 ± 0.0175	3.2259	3.701 ± 0.089
10^4	200	0.1	1	$\sqrt{2}$	141.54	140.74 ± 0.353	1983	1990.63 ± 16.89	0.0246	0.046 ± 0.011	−0.0611	-0.0713 ± 0.0202
10^4	200	0.1	0	$\sqrt{2}$	35.68	35.46 ± 0.17	726.76	732.39 ± 7.53	0.9959	1.0012 ± 0.012	0.8692	0.879 ± 0.035
10^4	200	0.1	−1	$\sqrt{2}$	12.16	11.35 ± 0.203	131.38	130.31 ± 6.059	1.697	1.747 ± 0.0827	3.907	4.076 ± 0.6277
10^4	200	0.2	1	$\sqrt{2}$	71.75	69.59 ± 0.15	924.70	925.01 ± 4.44	0.1576	0.185 ± 0.007	−0.2145	-0.201 ± 0.0135
10^4	200	0.2	0	$\sqrt{2}$	25.23	24.99 ± 0.082	363.38	376.21 ± 2.59	0.9953	0.9967 ± 0.0083	0.8692	0.8616 ± 0.0379
10^4	200	0.2	−1	$\sqrt{2}$	10.99	10.34 ± 0.027	101.30	102.31 ± 0.647	1.576	1.628 ± 0.01	3.2259	3.45 ± 0.069

Arrival = $\mathrm{Exp}(\lambda + \beta \cdot \sqrt{\lambda})$, Service = Log-Normal$(1/\lambda, 1/\lambda^2)$, Abandonment = Unif$(0, \frac{1}{\theta})$.

2.2. Numerical results for stationary setting

In this section we compare the skewness and kurtosis formulas with simulated examples of the truncated normal distribution. We see in Table 1 that our exact formulas do a great job of estimating the mean, variance, skewness, and kurtosis of the truncated normal distribution. We also provide confidence intervals and our explicit formulas lie within the given confidence intervals. Our 95% confidence intervals are obtained by performing 100 replications of the random vector of truncated normal random variables. To calculate our the confidence intervals for each cumulant we compute the standard deviation of each cumulant needed in the simulations, and multiply by 0.196, which is $1.96/\sqrt{100}$. Moreover, in Table 2 we compare our exact formulas with several queueing processes in heavy traffic with general distributions. We see that our formulas for the mean, variance, skewness, and kurtosis are good at estimating the performance of the queueing process in heavy traffic. The truncated normal approximation is less accurate than in Table 1 because the queueing process has finite rates and the queueing process is not exactly a truncated normal distribution. Nonetheless, the formulas give reasonable estimates for the behavior that are 1%–2% off from the actual values. We also provide confidence intervals in this case as well. Our 95% confidence intervals are also obtained by doing 100 simulations and by computing the standard deviation of each estimate in the simulations, and multiplying by 0.196, which is $1.96/\sqrt{100}$.

3. A truncated normal approximation for the $M_t/M_t/1 + M_t$ queue

In this section, we are inspired by the truncated normal distribution of the previous section and want to understand how a time varying truncated normal distribution might be useful as an approximation to the nonstationary single server queue with abandonment or the Markovian $M_t/M_t/1 + M_t$ queue. Since we are dealing with a Markovian queueing system, it is possible to express the queueing process in terms of time changed Poisson processes. By extending the work of [7] to queues with abandonment, we have the following stochastic integral representation for the queue length process as

$$Q(t) = Q(0) + \Pi_1\left(\int_0^t \lambda(s)ds\right) - \Pi_2\left(\int_0^t \mu(s)\cdot\{Q(s) > 0\}ds\right) - \Pi_3\left(\int_0^t \beta(s)\cdot Q(s)ds\right)$$

where $\Pi_i \equiv \{\Pi_i(t)|t \geq 0\}$ for $i = 1, 2, 3$ are independent standard (rate 1) Poisson processes. When the abandonment rate is zero, the work of [7] provides limit theorems for the sample path mean and variance of the queue length process. However, currently there are no limit theorems that include the possibility of abandonment for time varying queueing processes. In fact, applying the methods developed in [8] would require the rate functions of the single server queue to have Lipschitz continuous rate functions. However, the rate functions for the single server queue are not even continuous, since the service rate function depends on the queue is empty or not. Moreover, the functional central limit theorem approximations of [7] do not have to be diffusions and the uniform acceleration expansions of [12] are for bounded Markov chains. Thus, we need new methods to analyze these queueing processes and one potential method to develop approximations that are non-asymptotic is to use the functional Kolmogorov forward equations of the queueing process.

3.1. Functional forward equations

Like in the work of [11], we can derive the following *functional version* of the Kolmogorov forward equations for the $M_t/M_t/1+M_t$ queue, which are of the form

$$\begin{aligned}\overset{\bullet}{E}[f(Q)] &= \lambda\cdot E[f(Q+1) - f(Q)] \\ &\quad + \mu\cdot E[\{Q > 0\}\cdot(f(Q-1) - f(Q))] \\ &\quad + \beta\cdot E[Q\cdot(f(Q-1) - f(Q))]\end{aligned}$$

for all functions f that are well defined and integrable. Moreover, it is also possible to write the functional forward equations as

$$\begin{aligned}\overset{\bullet}{E}[f(Q)] &= \lambda\cdot E[f(Q+1) - f(Q)] \\ &\quad + \mu\cdot E[(Q\wedge 1)\cdot(f(Q-1) - f(Q))] \\ &\quad + \beta\cdot E[Q\cdot(f(Q-1) - f(Q))].\end{aligned}$$

Unlike the first representation, the service rate function is now a Lipschitz function of the queue length. This equivalence holds only because the queue length process is an integer valued process and the indicator function $\{Q > 0\}$ has the same value as the minimum function $Q\wedge 1$ on the integers. It is our experience that our methods work better for this second representation than with the indicator function representation.

For special cases of f such as the mean and variance, we can then obtain the following set of moment equations:

$$\begin{aligned}\overset{\bullet}{E}[Q] &= \lambda - \mu\cdot E[(Q\wedge 1)] - \beta\cdot E[Q] \\ \overset{\bullet}{\mathrm{Var}}[Q] &= \lambda + \mu\cdot E[(Q\wedge 1)] + \beta\cdot E[Q] \\ &\quad - 2\cdot\mu\cdot \mathrm{Cov}[Q, (Q\wedge 1)] - 2\cdot\beta\cdot\mathrm{Var}[Q].\end{aligned} \tag{3.5}$$

Although these equations describe the time varying dynamics of the queueing process, they are non-trivial to solve since the expectation terms depend on the queue length distribution beyond knowing the mean or variance. See for example [16] for more information regarding this distributional dependence of the mean and variance.

3.2. Truncated normal approximation

Inspired by the truncated normal distribution in the stationary setting, we apply a modified time varying truncated normal distribution to approximate the mean and variance of the single server queue with abandonment and time varying parameters. We define our modified truncated normal approximation of the queue length process at each fixed time point as the following random variable

$$Q \equiv q + \sqrt{\frac{v\cdot 2\cdot\pi}{\pi - 1}}\cdot\left(X^+ - \varphi(0)\right). \tag{3.6}$$

This approximation of the queue length leads us to our next theorem.

Theorem 3.1. *Suppose Q has the same distribution as Eq. (3.6), then we have that*

$$\begin{aligned}E[Q] &= q \\ \mathrm{Var}[Q] &= v \\ E[(Q\wedge 1)] &= q - \sqrt{\frac{\pi - 1}{2\cdot\pi\cdot v}}\cdot\left(\varphi(\chi) - \chi\cdot\overline{\Phi}(\chi)\right) \\ \mathrm{Cov}[Q, (Q\wedge 1)] &= \frac{v\cdot 2\cdot\pi}{\pi - 1}\cdot\big(\overline{\Phi}(\min(\chi, 0)) - \varphi(0) \\ &\quad\cdot\left(\varphi(\chi) - \chi\cdot\overline{\Phi}(\chi)\right)\big)\end{aligned}$$

where

$$\chi = \varphi(0) + \frac{(1-q)\cdot(\pi - 1)}{2\cdot\pi\cdot\sqrt{v}}.$$

Proof. For the mean of the queue length process, we have that

$$\begin{aligned}E[Q] &= E\left[q + \sqrt{\frac{v\cdot 2\cdot\pi}{\pi - 1}}\cdot\left(X^+ - \varphi(0)\right)\right] \\ &= q + \sqrt{\frac{v\cdot 2\cdot\pi}{\pi - 1}}\cdot\left(E\left[X^+\right] - \varphi(0)\right) \\ &= q + \sqrt{\frac{v\cdot 2\cdot\pi}{\pi - 1}}\cdot\left(E\left[X\cdot\{X\geq 0\}\right] - \varphi(0)\right) \\ &= q + \sqrt{\frac{v\cdot 2\cdot\pi}{\pi - 1}}\cdot\left(E\left[\delta_0(X)\right] - \varphi(0)\right) \\ &= q.\end{aligned}$$

Similar computations for the variance of the queue length process yield that

$$\begin{aligned}\mathrm{Var}[Q] &= \frac{v\cdot 2\cdot\pi}{\pi - 1}\cdot E[(X^+ - \varphi(0))^2] \\ &= \frac{v\cdot 2\cdot\pi}{\pi - 1}\cdot E[(X^2 - 1)\cdot\{X\geq 0\} + \{X\geq 0\} - \varphi(0)^2] \\ &= v.\end{aligned}$$

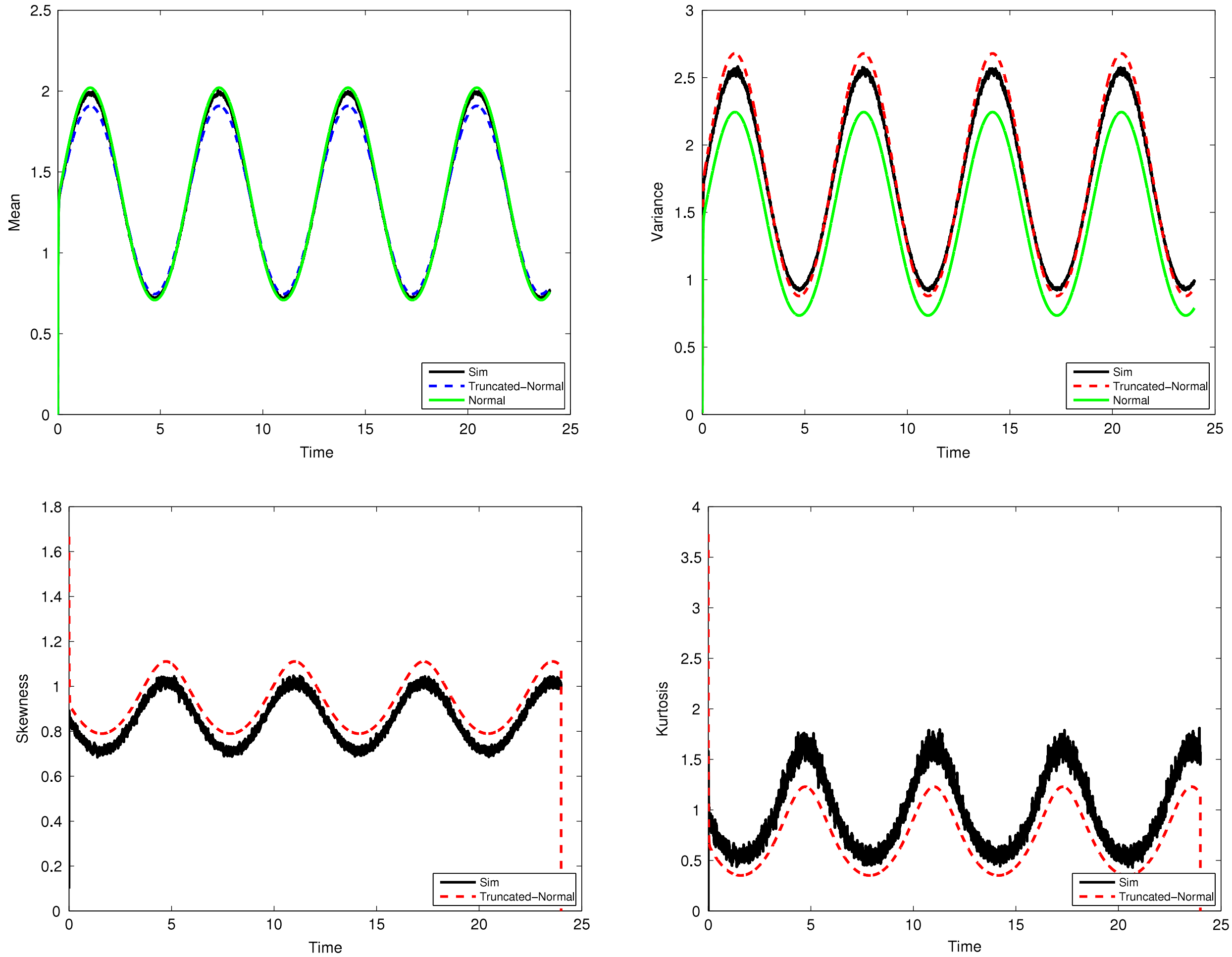

Fig. 1. $\lambda(t) = 100 + 40\sin(t),\ \mu = 100,\ \beta = 50,\ q(0) = 0,\ T = 24$. Mean Queue Length (Top Left). Variance of Queue Length (Top Right). Skewness of Queue Length (Bottom Left). Kurtosis of Queue Length (Bottom Right).

Lastly, for the minimum function we exploit the following identities

$$E[(Q \wedge 1)] = E[Q] - E[(Q-1)^+]$$
$$\mathrm{Cov}[Q, (Q \wedge 1)] = \mathrm{Var}[Q] - \mathrm{Cov}[Q, (Q-1)^+]$$

and solve for the functions $E[(Q-1)^+]$ and $\mathrm{Cov}[Q, (Q-1)^+]$. Computations similar to the ones that generate the mean and variance yield the following expressions.

$$E[(Q-1)^+] = \sqrt{\frac{\pi - 1}{2 \cdot \pi \cdot v}} \cdot \left(\varphi(\chi) - \chi \cdot \overline{\Phi}(\chi)\right)$$
$$\mathrm{Cov}[Q, (Q-1)^+] = \frac{v \cdot 2 \cdot \pi}{\pi - 1} \cdot \left(\overline{\Phi}(\min(\chi, 0)) - \varphi(0) \cdot \left(\varphi(\chi) - \chi \cdot \overline{\Phi}(\chi)\right)\right)$$

where

$$\chi = \varphi(0) + \frac{(1-q) \cdot (\pi - 1)}{2 \cdot \pi \cdot \sqrt{v}}. \quad \square \tag{3.7}$$

This theorem provides us with closed form expressions for the rate functions that appear in the functional forward equations of Eq. (3.5). Thus, using a time varying truncated normal distribution, we can develop accurate approximations for the mean and variance of the nonstationary queue length process, which has been neglected in the literature thus far. We can also compute various cumulants by using the formulas in the previous section. Since we have approximations for the mean and variance, we use these in the formulas for calculating the cumulants of the truncated normal distribution to approximate the skewness and kurtosis of the nonstationary queueing process.

3.3. *Numerical results for nonstationary setting*

On the top of Fig. 1 we see that the truncated normal distribution is approximating the mean and variance of the queue length process quite well. Moreover, we also compare the truncated normal distribution to that of the normal distribution with the same mean and variance and we see that the truncated normal distribution outperforms. On the bottom of Fig. 1 we plot the skewness and kurtosis of the time varying truncated normal approximation. It appears that it is overestimating the skewness a bit and it is approximating the kurtosis quite well. There is no need to calculate the skewness and kurtosis directly since we can use the formulas of Section 2 directly. We note that for large values of the mean and variance of the queue length i.e. $q, v > 10$, we can use the same mean and variance generated from the truncated normal approximation. However, when the queue length is smaller we have to adjust the mean and variance formulas, which can be done using the *fzero* function in Matlab [13]. If we were to use a normal distribution, the skewness and kurtosis approximations have the value zero, which is clearly not an accurate estimate of the true skewness and kurtosis simulations. We should expect this difference as the approximation for the skewness and kurtosis is very sensitive to small changes in the mean and variance when the mean and variance are near zero. Moreover, our approximations of the skewness and kurtosis are based on our approximations of the mean and variance, which have an error of 1%–2% and 5%–10% respectively. This error in the mean and variance can yield an error of 10%–15% in the skewness and kurtosis approximations. We believe that this approach can be extended to single server loss models and nonstationary phase type models by expanding the state space and the dimensionality of the approximations.

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