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An analysis of nonstationary coupled queues

Jamol Pender[1]

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Abstract We consider a two dimensional time varying tandem queue with coupled processors. We assume that jobs arrive to the first station as a non-homogeneous Poisson process. When each queue is non-empty, jobs are processed separately like an ordinary tandem queue. However, if one of the processors is empty, then the total service capacity is given to the other processor. This problem has been analyzed in the constant rate case by leveraging Riemann Hilbert theory and two dimensional generating functions. Since we are considering time varying arrival rates, generating functions cannot be used as easily. Thus, we choose to exploit the functional Kolmogorov forward equations (FKFE) for the two dimensional queueing process. In order to leverage the FKFE, it is necessary to approximate the queueing distribution in order to compute the relevant expectations and covariance terms. To this end, we expand our two dimensional Markovian queueing process in terms of a two dimensional polynomial chaos expansion using the Hermite polynomials as basis elements. Truncating the polynomial chaos expansion at a finite order induces an approximate distribution that is close to the original stochastic process. Using this truncated expansion as a surrogate distribution, we can accurately estimate probabilistic quantities of the two dimensional queueing process such as the mean, variance, and probability that each queue is empty.

Keywords Coupled processors · Abandonment · Dynamical systems · Discontinuous coefficients · Time-varying rates · Polynomial chaos · Hermite polynomials

✉ Jamol Pender
jjp274@cornell.edu

1 School of Operations Research and Information Engineering, Cornell University, Ithaca, USA

1 Introduction

In this paper we consider a tandem queueing model consisting of two, yet coupled, stations with the possibility of abandonment. Jobs arrive at the first station according to a non-homogeneous Poisson process. If the job receives service at the first station, it will proceed to the second station. However, we also include the possibility for a job to abandon the system if it does not begin to receive service within an independent, but exponentially distributed amount of time. Moreover, if the job receives service from the first station and moves to the second station, the job can also abandon from the second station if it does not receive service within another independent exponentially distributed amount of time. Finally, if the job does receive service in the second station, then it leaves the system entirely. When both stations are nonempty, a given proportion of the capacity is allocated to station 1, and the remaining proportion is allocated to station 2. However, if one of the stations is empty, the total service capacity of the stations is allocated to the other station.

1.1 Applications of queueing model in telecommunications

The queueing model with coupled processors that we study in this paper is an appropriate stochastic model for a variety of applications in telecommunications and beyond. One such application that is noted by Resing and Ormeci [25] is data transfer in bidirectional cable and data networks. An example of a bidirectional cable or data network is when a user sends out data (for example, an e-mail message) while simultaneously receiving other data (for example, downloading music). It is common for the user in cable networks to download more data and information than uploading. If the two processors for uploading and downloading are coupled, then the uploading direction can assist the downloading direction to speed up the processing. In computer science and the electrical engineering literature this can be called cycle stealing without switching costs.

Another application for this coupled processor model is modeling of cellular networks. For instance, one can consider a system consisting of two neighboring cellular base stations. A base station will serve its associated users, however, if the base station is idle when there are no active users associated with it, then it can assist the neighboring base station with serving its customers. This is very common in cellular communication networks. For example, when there are two base stations, one of which is heavily loaded and the other which is lightly loaded, a coupled system can ensure that no capacity is wasted via idle base stations.

An additional application of where our model is relevant is in the study of shared resources in bandwidth sharing of data flows. Bandwidth sharing in data flows usually are modeled using generalized processor sharing queues (GPS queues). Unlike the priority discipline, the GPS scheme allows for service differentiation and priorization while never neglecting any class of service. GPS queues are typically coupled when there is excess capacity that is shared among all the resources. This coupling speeds up throughput when some of the capacity would otherwise be idle and wasted.

The last application, is seen in the work of Andradottir et al. [2]. Although not motivated in telecommunications, they use the coupled processor model to analyze the throughput of assembly lines. Suppose that the assembly line has only two stations. When one station is empty, then the other station can help decrease the processing time of the non-empty station, thereby increasing the throughput of the entire assembly line.

1.2 Previous literature

The study of coupled processors is not a new topic. In a seminal paper, Fayolle and Iasnogorodski [9] were the first to consider coupled processors. In their paper, they analyzed two coupled servers in parallel with exponential service times. To gain quantitative insights about the queueing process, they derived a solution for the generating function of the stationary distribution of the Markov process describing the number of jobs in both queues, exploiting the theory of Riemann-Hilbert boundary value problems. Cohen and Boxma [7] extended the model by analyzing the coupled processor model for the case of generally distributed service times and Boxma and Ivanovs [5] study the workload processes under generally distributed service times and a Levy input process. Konheim et al. [12] use a novel uniformization method to determine the generating function of the queue length process of both queues in a parallel and not a tandem setting. Their method however, only considers the case when the queues have both identical arrival and service rates. Blanc et al. [4] study the tandem queueing model using boundary value theory with general service times at the first station, but they do not consider coupled processors. A power series method has been used to study the coupled processor in Blanc [3]. Lastly, in the computer science literature, Osogami et al. [18] study the coupled processor with switching costs and analyze cycle stealing problems in computer networks.

Although most of the past research on this problem has relied heavily on the study of two dimensional generating functions, authors such as Knessl and Morrison [10] study the asymptotics of the coupled processor in heavy traffic. They are able to show that in the heavy traffic regime, one can compute performance measures like the mean and the steady state distribution by studying an elliptic partial differential equation. Moreover, using the singular perturbation methods, Knessl [11] is able to derive asymptotic formulas for first passage times of the busy period of each queue. However, the study of these systems has been limited to constant rate dynamics and has not included the possibility that jobs might abandon the queue if their job is not completed in a satisfactory amount of time.

In this paper, we attempt to study the dynamics of the coupled processor with the new features of time varying arrival rates and the possibility of jobs abandoning. The motivation for analyzing these important features is that they are often neglected in the literature since they often make estimating the queueing behavior intractable. Even without these new features, coupled processor queueing models are very difficult to analyze, thus we must find a method that can tackle these additional complex features. In this paper, we specifically choose to analyze the tandem model that was considered in Leeuwaarden and Resing [13]. However, our method easily extends to other Markovian models such as the one considered by Wright [27], which considers two parallel M/M/1 queues with unique arrival rates instead of a tandem model.

Contrary to the current literature that uses generating functions, we cannot use this tool because of the time varying arrival process. To this end, we exploit the functional Kolmogorov forward equations of the Markovian coupled processor. The functional forward equations describe the time derivative of functions of the queue length processes. The main difficulty of using the functional forward equations is that they are not a closed system because of the reflection at zero and the boundary conditions of the coupled system. Since the forward equations are not a closed dynamical system, this implies that we need to understand the a priori distribution of the queue length process in order to compute the time derivative of various functions of the queue length processes. In order to combat this difficulty, we propose to approximate the distribution of the queue length process with a two dimensional Gaussian surrogate distribution. This reduces our estimation of the mean and variance of the queue length process to computing expectations and covariance terms with respect to a two dimensional Gaussian measure. Although many of the resulting expectations and covariance terms can be written in terms of an infinite series of Hermite polynomials, we will show that using the first two or three terms is sufficient for our approximation of the functional forward equations.

1.3 Contributions

In analyzing this model, we make the following key contributions:

- We are the first to analyze this model with the added features of time varying arrival rates and the possibility of abandonment, which are important features of processors.
- We show that by using polynomial chaos expansions with Hermite basis functions, we are able to estimate the mean, variance, and probability of emptiness for our time varying coupled queueing process.
- Develop a numerical method that only requires the solution of five differential equations for estimating the most important performance measures.

1.4 Organization of the paper

The rest of the paper is organized as follows. Section 2 describes the queueing model, its sample path representation, and its associated Kolmogorov functional forward equations. In Sect. 3, we give a motivation for using the Hermite polynomial expansion for the queue length process and we show how the functional forward equations can be used to construct Hermite polynomial approximations (DMA and GVA) for the mean, variance, and probability that each queue is empty. We also give a representative numerical example for the paper in this section. In Sec. 4, we conclude and give final remarks. Lastly, we compute all our expectations and covariances in Sect. 5 or the Appendix.

2 Stochastic analysis of the coupled processor

2.1 Sample path construction of queueing process via poisson random measures

In this section, we define and characterize the stochastic queueing process under consideration. Using Poisson random measures to construct the sample paths of the queueing process, similar to the work of Mandelbaum et al. [14], we can show that the coupled processor queueing system $\{Q(t)|t \geq 0\} \equiv (Q_1(t), Q_2(t))$ can be represented by the following stochastic integral equations:

$$\begin{aligned}
Q_1(t) = Q_1(0) &+ \Pi_1\left(\int_0^t \lambda(s)ds\right) \\
&-\Pi_2\left(\int_0^t \mu_1 \cdot \{Q_1 > 0\} \cdot \{Q_2 > 0\} ds\right) \quad (2.1)\\
&-\Pi_3\left(\int_0^t (\mu_1+\mu_2) \cdot \{Q_1 > 0\} \cdot \{Q_2 \leq 0\} ds\right)\\
&-\Pi_4\left(\int_0^t \beta_1 \cdot Q_1(s) ds\right) \quad (2.2)
\end{aligned}$$

$$\begin{aligned}
Q_2(t) = Q_2(0) &+ \Pi_2\left(\int_0^t \mu_1 \cdot \{Q_1 > 0\} \cdot \{Q_2 > 0\} ds\right) \quad (2.3)\\
&+\Pi_3\left(\int_0^t (\mu_1+\mu_2) \cdot \{Q_1 > 0\} \cdot \{Q_2 \leq 0\} ds\right)\\
&-\Pi_5\left(\int_0^t \beta_2 \cdot Q_2(s) ds\right) \quad (2.4)\\
&-\Pi_6\left(\int_0^t \mu_2 \cdot \{Q_1 > 0\} \cdot \{Q_2 > 0\} ds\right) \quad (2.5)\\
&-\Pi_7\left(\int_0^t (\mu_1+\mu_2) \cdot \{Q_1 \leq 0\} \cdot \{Q_2 > 0\} ds\right) \quad (2.6)
\end{aligned}$$

where each $(\Pi_i) \equiv \{(\Pi_i)|t \geq 0\}$ are i.i.d. standard (rate 1) Poisson processes. Each of the processes (Π_i) contain probabilistic information about the queueing process. Poisson random measures with intensity function $g(t)$ are defined by two important properties. The first property is that the number of arrivals in non-overlapping intervals are statistically independent. The second property is that the number of arrivals in an interval $(s, t]$ follows the Poisson distribution i.e.

$$P(A(s,t] = n) = \frac{\left(\int_s^t g(u)du\right)^n}{n!} \exp\left(-\int_s^t g(u)du\right) \quad (2.7)$$

for all positive integers n. However, another way to view Poisson random measures is through the method of time change. For instance a deterministic time change for the process Π_1 transforms it into a non-homogeneous Poisson arrival process with rate $\lambda(t)$ that counts the number of customer arrivals in the interval (0,T]. A random time change for Π_2 yields a process that counts the number of service departures in the interval (0,T] from Q_1 to Q_2 when both queues are non-empty. A random time change for Π_3 counts the number of service departures in the interval (0,T] from Q_1 to Q_2 when the first queue is non-empty, but the second queue is empty. A random time change for Π_4 counts the number of abandonment in the interval (0,T] from Q_1. A random time change for Π_5 counts the number of abandonments in the interval (0,T] from Q_2. A random time change for

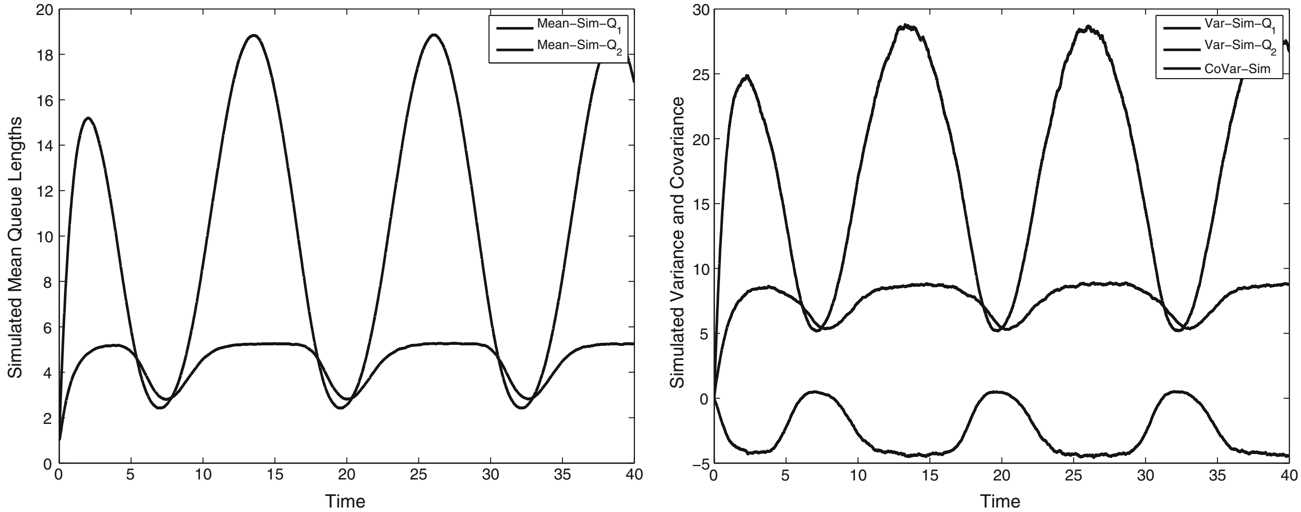

Fig. 1 Simulated mean of both queues (*Left*). Simulated variance and covariance of both queues (*Right*)

Π_6 counts the number of service departures in the interval (0,T] from Q_2 when both queues are non-empty. Lastly, a random time change for Π_7 counts the number of service departures in the interval (0,T] from Q_2 while the first queue is empty.

There is one important observation to make about the sample path construction of the queue length processes. The rate functions of the stochastic representation are not Lipschitz continuous. This is one of many reasons that precludes us from applying the standard fluid and diffusion limit theorem results of Mandelbaum et al. [14]. Thus, there are currently no fluid and diffusion limit theorems that have been proved for this particular process. Thus, our approach to derive closed form approximations for our queueing process, will be to use the functional forward equations for the queue length process and construct a surrogate distribution to *close* the forward equations.

To get a better understanding of the coupled processor, we simulate the queueing process. For the example that we will consider throughout the paper, we assume that the arrival rate is $20 + 10 \cdot \sin(t)$, the service rate for the first queue is $\mu_1 = 10$, the service rate for the second queue is $\mu_2 = 5$, and the abandonment rates for each queue is $\beta_1 = \beta_2 = 1$. We also simulate the system over the time interval of (0,40], with a time step of $\Delta t = .001$ for 10,000 sample paths.

On the left of Fig. 1 we simulate the mean of both queues. Due to the time varying arrival rate, we see that the dynamics of the mean queue length process is also time varying. On the right of Fig. 1, we simulate the variance and covariance of both queues. Similar to the mean queue length, the variance and covariance are also time varying functions of time. One other observation of the right of Fig. 1, we see that the covariance peaks when the queue length of the first and second queue are minimal. When observing Fig. 2, this suggests that the covariance of the coupled processor is larger when the probability that the queues are coupled is larger. This is behavior that one cannot observe in the stationary case, when the mean queue length reaches a steady state and does not vary over time.

2.2 Forward equations for the Tandem model

We start this section with a derivation of the *functional* Kolmogorov forward equations for our tandem queueing model. We will then use the functional version of these equations to derive our approximation for the mean, variance, and probability of emptiness for our coupled processor queueing model. First we let f be any integrable, real valued function on the two dimensional state space of our tandem queueing model, then we have

$$\begin{aligned}
&\dot{E}[f(Q_1, Q_2)] \\
&\quad = \lambda_1(t) \cdot E\left[f(Q_1 + 1, Q_2) - f(Q_1, Q_2)\right] \\
&\qquad + \mu_1 \cdot E\left[(f(Q_1 - 1, Q_2) - f(Q_1, Q_2 \right. \\
&\qquad \left. + 1)) \cdot \{Q_1 \geq 0\} \cdot \{Q_2 \geq 0\}\right] \\
&\qquad + \mu_2 \cdot E\left[(f(Q_1, Q_2 - 1) - f(Q_1, Q_2)) \right. \\
&\qquad \left. \cdot \{Q_1 \geq 0\} \cdot \{Q_2 \geq 0\}\right] \\
&\qquad + (\mu_1 + \mu_2) \cdot E\left[(f(Q_1 - 1, Q_2 + 1) \right. \\
&\qquad \left. - f(Q_1, Q_2)) \cdot \{Q_1 \geq 0\} \cdot \{Q_2 \leq 0\}\right] \\
&\qquad + (\mu_1 + \mu_2) \cdot E\left[(f(Q_1, Q_2 - 1) \right. \\
&\qquad \left. - f(Q_1, Q_2)) \cdot \{Q_1 \leq 0\} \cdot \{Q_2 \geq 0\}\right] \\
&\qquad + \beta_1 \cdot E\left[(f(Q_1 - 1, Q_2) - f(Q_1, Q_2)) \cdot Q_1\right] \\
&\qquad + \beta_2 \cdot E\left[(f(Q_1, Q_2 - 1) - f(Q_1, Q_2)) \cdot Q_2\right]
\end{aligned}$$

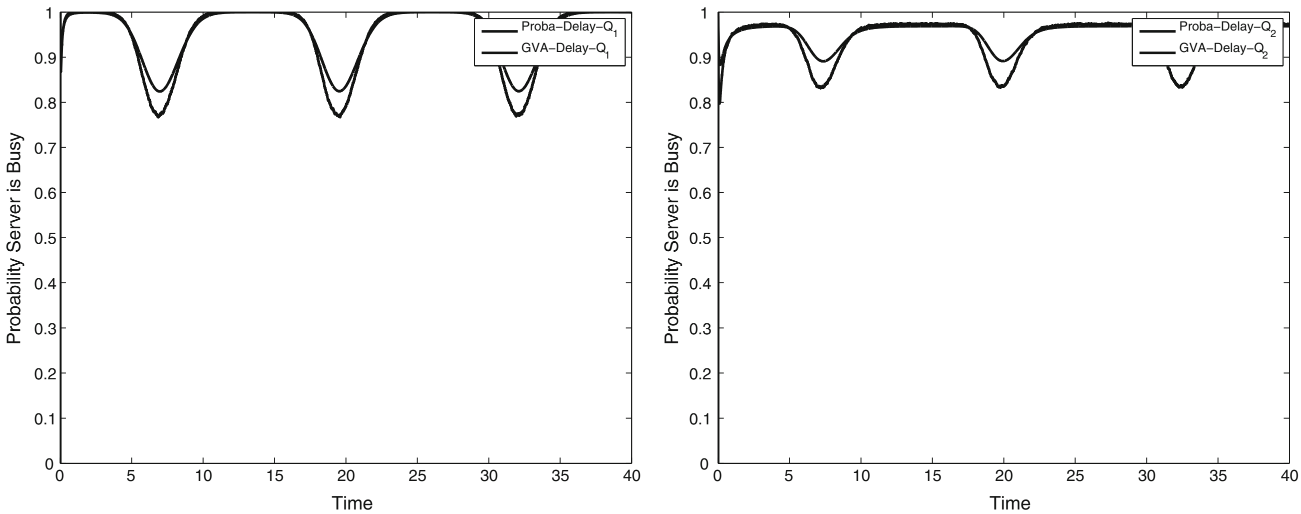

Fig. 2 Simulated probability of emptiness of queue 1 (*Left*). Simulated probability of emptiness of queue 2 (*Right*)

where $\dot{E}[f(Q_1, Q_2)]$ is defined as

$$\dot{E}[f(Q_1, Q_2)] \equiv \frac{d}{dt} E[f(Q_1, Q_2)], \tag{2.8}$$

which is the time derivative of the expectation of the function $f(Q_1, Q_2)$. A rigorous derivation of these equations can be found in Engblom et al. [8] in the one-dimensional setting and in Massey and Pender [15–17] for the multi-dimensional setting.

If we focus instead on the marginal distribution of the forward equations for Q_1 i.e. $f(x, y) = g(x)$, we have the following functional forward equations for Q_1

$$\begin{aligned}\dot{E}[g(Q_1)] = \lambda_1(t) \cdot E\left[g(Q_1 + 1) - g(Q_1)\right] \\ + \mu_1 \cdot E\left[(g(Q_1 - 1) - g(Q_1)) \cdot \{Q_1 \geq 0\}\right. \\ \left. \cdot \{Q_2 \geq 0\}\right] \\ + (\mu_1 + \mu_2) \cdot E\left[(g(Q_1 - 1)\right. \\ \left. - g(Q_1)) \cdot \{Q_1 \geq 0\} \cdot \{Q_2 \leq 0\}\right] \\ + \beta_1 \cdot E\left[(g(Q_1 - 1) - g(Q_1)) \cdot Q_1\right].\end{aligned}$$

Similarly, if we focus instead on the marginal distribution for the forward equations for Q_2 i.e. $f(x, y) = w(y)$, then we have the following functional forward equations for Q_2

$$\begin{aligned}\dot{E}[w(Q_2)] = \mu_2 \cdot E\left[(w(Q_2 - 1) - w(Q_2))\right. \\ \left. \cdot \{Q_1 \geq 0\} \cdot \{Q_2 \geq 0\}\right] \\ + (\mu_1 + \mu_2) \cdot E\left[(w(Q_2 - 1)\right. \\ \left. - w(Q_2)) \cdot \{Q_1 \leq 0\} \cdot \{Q_2 \geq 0\}\right] \\ + \beta_2 \cdot E\left[(w(Q_2 - 1) - w(, Q_2)) \cdot Q_2\right].\end{aligned}$$

If we restrict to understanding the time dependent behavior for the mean of our queue length processes, then we have the following equations for the time derivatives of $E[Q_1], E[Q_2]$

$$\begin{aligned}\dot{E}[Q_1] = \lambda_1(t) - \mu_1 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\ - (\mu_1 + \mu_2) \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 \leq 0\}\right] \\ - \beta_1 \cdot E[Q_1]\end{aligned}$$

$$\begin{aligned}\dot{E}[Q_2] = \mu_1 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] + (\mu_1 + \mu_2) \\ \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 \leq 0\}\right] \\ - \mu_2 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\ - (\mu_1 + \mu_2) \cdot E\left[\{Q_1 \leq 0\} \cdot \{Q_2 > 0\}\right] \\ - \beta_2 \cdot E[Q_2]\end{aligned}$$

Moreover, for the variances and covariance i.e. $\mathrm{Var}[Q_1]$, $\mathrm{Var}[Q_2]$, $\mathrm{Cov}[Q_1, Q_2]$ we have that

$$\begin{aligned}\dot{\mathrm{Var}}[Q_1] = \lambda_1(t) + \mu_1 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\ + (\mu_1 + \mu_2) \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 \leq 0\}\right] \\ + \beta_1 \cdot E[Q_1] \\ -2 \cdot \mu_1 \cdot \mathrm{Cov}\left[Q_1, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\ -2 \cdot (\mu_1 + \mu_2) \cdot \mathrm{Cov}\left[Q_1, \{Q_1 > 0\}\right. \\ \left. \cdot \{Q_2 \leq 0\}\right] - 2 \cdot \beta_1 \cdot \mathrm{Cov}\left[Q_1, Q_1\right]\end{aligned}$$

$$\begin{aligned}\dot{\mathrm{Var}}[Q_2] = \mu_1 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] + (\mu_1 + \mu_2) \\ \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 \leq 0\}\right] \\ + \mu_2 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] + (\mu_1 + \mu_2) \\ \cdot E\left[\{Q_1 \leq 0\} \cdot \{Q_2 > 0\}\right] + \beta_2 \cdot E[Q_2] \\ + 2 \cdot \mu_1 \cdot \mathrm{Cov}\left[Q_2, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\ - 2 \cdot \mu_2 \cdot \mathrm{Cov}\left[Q_2, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right]\end{aligned}$$

$$
\begin{aligned}
&+ 2 \cdot (\mu_1 + \mu_2) \cdot \mathrm{Cov}\,[Q_2, \{Q_1 \le 0\} \\
&\cdot \{Q_2 > 0\}] \\
&- 2 \cdot (\mu_1 + \mu_2) \cdot \mathrm{Cov}\,[Q_2, \{Q_1 \le 0\} \\
&\cdot \{Q_2 > 0\}] - 2 \cdot \beta_2 \cdot \mathrm{Cov}\,[Q_2, Q_2]
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\mathrm{C}}\mathrm{ov}[Q_1, Q_2] \\
&= -\mu_1 \cdot E\,[\{Q_1 > 0\} \cdot \{Q_2 > 0\}] - (\mu_1 + \mu_2) \\
&\cdot E\,\, [\{Q_1 > 0\} \cdot \{Q_2 \le 0\}] \\
&+ \mu_1 \cdot \mathrm{Cov}\,[Q_1, \{Q_1 > 0\} \cdot \{Q_2 > 0\}] \\
&- \mu_1 \cdot \mathrm{Cov}\,[Q_2, \{Q_1 > 0\} \cdot \{Q_2 > 0\}] \\
&- \mu_2 \cdot \mathrm{Cov}\,[Q_1, \{Q_1 > 0\} \cdot \{Q_2 > 0\}] \\
&- (\mu_1 + \mu_2) \cdot \mathrm{Cov}\,[Q_1, \{Q_1 > 0\} \cdot \{Q_2 \le 0\}] \\
&- (\mu_1 + \mu_2) \cdot \mathrm{Cov}\,[Q_2, \{Q_1 > 0\} \cdot \{Q_2 \le 0\}] \\
&- (\mu_1 + \mu_2) \cdot \mathrm{Cov}\,[Q_1, \{Q_1 \le 0\} \cdot \{Q_2 > 0\}] \\
&- (\beta_1 + \beta_2) \cdot \mathrm{Cov}\,[Q_1, Q_2]\,.
\end{aligned}
$$

3 Polynomial chaos expansions

In this section, we provide some background information on the expansion methods that follow. Our method is inspired by the well known Cameron-Martin theorem [6] which shows that any square integrable random variable $\mathcal{X}$ can be represented by a series expansion as follows:

$$\mathcal{X}(\xi) = \sum_{j=0}^{\infty} q_j \cdot \Psi_j(\xi) \tag{3.1}$$

where q_j are coefficients that are to be determined and the $\Psi_j(\xi)$ are orthogonal functions of the ξ random variables, which form a complete orthonormal polynomial basis. In fact the orthogonal functions $\Psi_j(\xi)$ satisfy the following properties

$$
\begin{aligned}
\Psi_0(\xi) &= 1 \\
E[\Psi_j(\xi)] &= 0, \quad \text{for } j \ge 1 \\
E[\Psi_j(\xi) \cdot \Psi_k(\xi)] &= 0, \quad \text{for } j \ne k
\end{aligned}
$$

Thus, the Cameron-Martin theorem [6] implies that any arbitrary random variable $\mathcal{X}(\xi)$, which is square integrable, can be represented by an infinite number of random variables with a particular distribution. To give the reader more insight into how this works we will give a simple explanation below. Let X be a random variable with cdf F_X. Let us also define U as a uniform random variable on [0,1]. We have that as long as the distribution of X is continuous that its cdf will be invertible i.e.

$$X = F_X^{-1}(U). \tag{3.2}$$

Moreover, if Z is an arbitrary continuous random variable with cdf F_z, then we have that

$$X = F_X^{-1}(F_z(Z)). \tag{3.3}$$

Thus, we can represent random variables in term of other random variables that are easier to manipulate. In this paper we choose to use the Hermite polynomials and Gaussian distribution since they are easy to manipulate and are used to perform difficult calculations. This type of chaos expansion has been considered by many authors in applied mathematics such as Ogura [19], Askey and Wilson [1], and Xiu and Karniadakis [28]. Unfortunately a perfect approximation of the random variable $\mathcal{X}(\xi)$ might require an infinite number of basis elements. However, one way to accurately approximate the random variable $\mathcal{X}(\xi)$ is to truncate the infinite series expansion at a finite order. Approximating $\mathcal{X}(\xi)$ with polynomials with degree less than or equal to n results in the following approximation for $\mathcal{X}(\xi)$

$$\mathcal{X}(\xi) \approx \sum_{j=0}^{n} q_j \cdot \Psi_j(\xi). \tag{3.4}$$

Now to extend the polynomial chaos expansion from random variables to stochastic processes, we need to make the coefficients q_j functions of time i.e.

$$\mathcal{X}(t, \xi) \approx \sum_{j=0}^{n} q_j(t) \cdot \Psi_j(\xi). \tag{3.5}$$

In the remainder of the paper, we will focus our expansion efforts on the Hermite polynomials, which are orthogonal with respect to the Gaussian measure. This means that we will let $\Psi_j(\xi) = H_j(\xi)$ where ξ is a standard Gaussian random variable and $H_j(\cdot)$ is the j^{th} Hermite polynomial. We will show how to use the Hermite polynomial chaos expansion to approximate the dynamics of the coupled processor and many of its performance measures.

3.1 Motivation for Hermite expansion

Suppose that the coupled processor $Q = (Q_1, Q_2)$ is a square integrable stochastic process Q on a bounded time interval [0,T]. The fact that the coupled processor is square integrable easily follows from the fact that the coupled processor is bounded above by the initial number of jobs and the nonstationary arrival rate, which we assume to have integrable rates. Since Q is square integrable, we can write Q as a weighted sum of orthogonal Hermite polynomials i.e.

$$Q = \sum_{n=0}^{\infty} q_n \cdot H_n(Y). \tag{3.6}$$

The series representation converges in the Hilbert space $L^2[0, \infty)$ if and only if

$$\|Q\|_{L^2}^2 = \sum_{n=0}^{\infty} n! \cdot \|q_n\|^2 < \infty. \tag{3.7}$$

One motivating reason why we use Hermite polynomials is that they are orthogonal with respect to the Gaussian measure, which is infinitely differentiable. Since many of the functions that appear in queueing theory are discontinuous or not differentiable (especially indicator functions), it is clear that integrating these functions or taking expectations with respect to the Gaussian measure makes the subsequent functions smooth.

Since our queue length process is square integrable by the standard Markovian service network assumptions, we have hope of applying the Hermite polynomial expansion. However, in order to avoid the infinite set of Hermite polynomials needed to approximate our functions with arbitrary precision, we choose to truncate the series expansion at a finite number of terms. This truncation is analogous to projecting our stochastic process onto a finite dimensional space of polynomial basis elements. For the first and second truncations, we respectively get the following approximations

$$Q \approx q_0^\alpha \cdot H_0(Y)$$
$$Q \approx q_0^\alpha \cdot H_0(Y) + q_1^\alpha \cdot H_1(Y).$$

These truncated Hermite series expansions motivate the following approximations for our stochastic queue length process.

3.2 First order: deterministic mean approximation (DMA)

Like in the work of Massey and Pender [17] and Pender [20], our first approximation will use the first Hermite polynomial for each queue length process as an approximate distribution of the queueing process. We call this the *Deterministic Mean Approximation* since we assume $\{q \equiv (q_1(t), q_2(t)) | t \geq 0\}$ is a deterministic process that approximates the queueing process.

Theorem 3.1 *If we let* (q_1, q_2) *replace* (Q_1, Q_2) *in the Kolmogorov forward equation for the distribution of* Q*, then* q *solves the resulting autonomous, two-dimensional, dynamical system*

$$\overset{\bullet}{q}_1 = \lambda_1 - \mu_1 \cdot \{q_1 > 0\} \cdot \{q_2 > 0\} - (\mu_1 + \mu_2) \cdot \{q_1 > 0\} \cdot \{q_2 \leq 0\} - \beta_1 \cdot q_1 \tag{3.8}$$

$$\begin{aligned}\overset{\bullet}{q}_2 = {} & \mu_1 \cdot \{q_1 > 0\} \cdot \{q_2 > 0\} + (\mu_1 + \mu_2) \cdot \{q_1 > 0\} \cdot \{q_2 \leq 0\} \\ & - \mu_2 \cdot \{q_1 > 0\} \cdot \{q_2 > 0\} \\ & - (\mu_1 + \mu_2) \cdot \{q_1 \leq 0\} \cdot \{q_2 > 0\} - \beta_2 \cdot q_2.\end{aligned} \tag{3.9}$$

Proof For the first queue length process we have that

$$\begin{aligned}\overset{\bullet}{E}[Q_1] &= \lambda_1(t) - \mu_1 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\ &\quad - (\mu_1 + \mu_2) \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 \leq 0\}\right] \\ &\quad - \beta_1 \cdot E[Q_1] \\ \overset{\bullet}{E}[q_1] &= \lambda_1(t) - \mu_1 \cdot E\left[\{q_1 > 0\} \cdot \{q_2 > 0\}\right] \\ &\quad - (\mu_1 + \mu_2) \cdot E\left[\{q_1 > 0\} \cdot \{q_2 \leq 0\}\right] \\ &\quad - \beta_1 \cdot E[q_1] \\ \overset{\bullet}{q}_1 &= \lambda_1(t) - \mu_1 \cdot \{q_1 > 0\} \cdot \{q_2 > 0\} \\ &\quad - (\mu_1 + \mu_2) \cdot \{q_1 > 0\} \cdot \{q_2 \leq 0\} - \beta_1 \cdot q_1\end{aligned}$$

and for the second queue length process we have that

$$\begin{aligned}\overset{\bullet}{E}[Q_2] &= \mu_1 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] + (\mu_1 + \mu_2) \\ &\quad \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 \leq 0\}\right] \\ &\quad - \mu_2 \cdot E\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] - (\mu_1 + \mu_2) \\ &\quad \cdot E\left[\{Q_1 \leq 0\} \cdot \{Q_2 > 0\}\right] - \beta_2 \cdot E[Q_2] \\ \overset{\bullet}{E}[q_2] &= \mu_1 \cdot E\left[\{q_1 > 0\} \cdot \{q_2 > 0\}\right] + (\mu_1 + \mu_2) \\ &\quad \cdot E\left[\{q_1 > 0\} \cdot \{q_2 \leq 0\}\right] \\ &\quad - \mu_2 \cdot E\left[\{q_1 > 0\} \cdot \{q_2 > 0\}\right] - (\mu_1 + \mu_2) \\ &\quad \cdot E\left[\{q_1 \leq 0\} \cdot \{q_2 > 0\}\right] - \beta_2 \cdot E[q_2] \\ \overset{\bullet}{q}_2 &= \mu_1 \cdot \{q_1 > 0\} \cdot \{q_2 > 0\} + (\mu_1 + \mu_2) \cdot \{q_1 > 0\} \\ &\quad \cdot \{q_2 \leq 0\} \\ &\quad - \mu_2 \cdot \{q_1 > 0\} \cdot \{q_2 > 0\} - (\mu_1 + \mu_2) \cdot \{q_1 \leq 0\} \\ &\quad \cdot \{q_2 > 0\} - \beta_2 \cdot q_2.\end{aligned}$$

This completes the proof. □

It is clear that the DMA yields the same equation as one would derive from a heuristic fluid limit argument. Thus, we can view DMA as an one dimensional projection onto the deterministic functions $q_1(t), q_2(t)$. In Fig. 3, we see that the DMA is doing a good job of approximating the mean queue length of the first queue. However, we see that the DMA does not do a good job of approximating the mean queue length of the second queue. This is because although the first queue has a time varying arrival rate, the arrival rate of the second queue is not time varying since it depends on the service rate of the first queue. Moreover, the service rate of the first queue does not change because the queue length of the first queue is always positive and, it would only change if the queue length were zero.

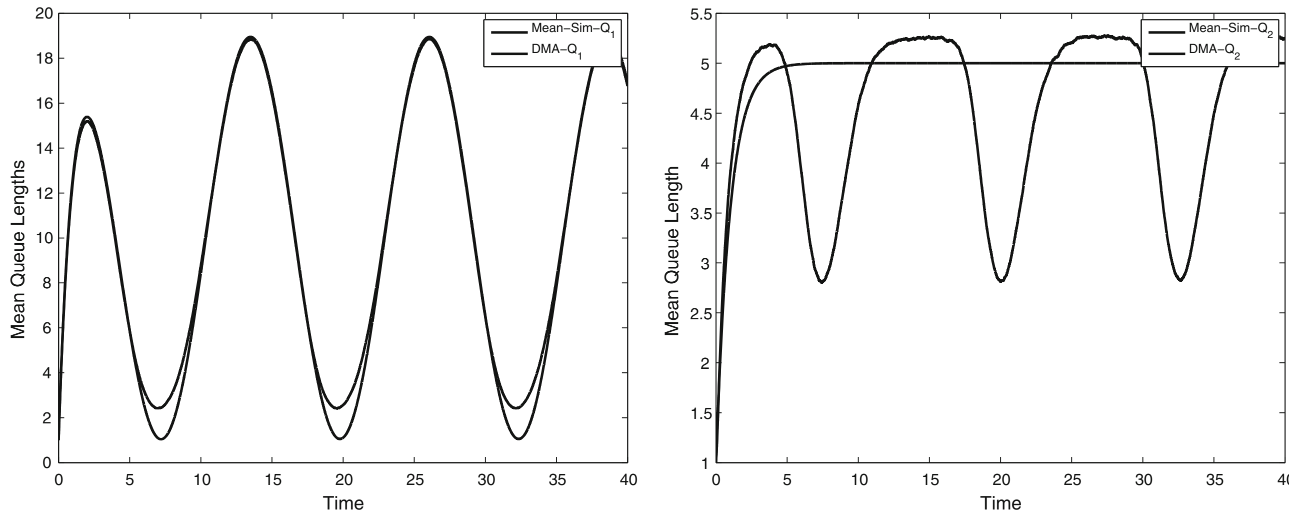

Fig. 3 Simulated mean and variance (*Left*). Simulated Skewness and Kurtosis (*Right*)

Now that we see that the mean queue length for the second queue is not well estimated by the DMA, this leads us to explore further polynomial expansions for the queue length process. Since DMA is a deterministic and non-random approximation for the mean queue length, DMA implicitly assumes that the variance and higher cumulant moments are equal to zero. Thus, if we want to appropriately model other moments such as the variance and covariance, we must add an additional term to the Hermite polynomial queueing expansion. The additional polynomials will add some randomness to the queue length approximation and not implicitly assume that the variance and covariance of the queue length processes are zero.

3.3 Second order: Gaussian variance approximation (GVA)

In this section we extend the DMA by adding an additional polynomial to the approximating queue length process. By adding an additional polynomial, we approximate the dynamics of the mean and variance of Q by a random process $Q \equiv \{Q(t) | t \geq 0\}$ such that

$$Q \overset{d}{=} \mathcal{N}(q, \Sigma) \tag{3.10}$$

where

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad and \quad \Sigma = \begin{pmatrix} v_1 & v_3 \\ v_3 & v_2 \end{pmatrix} \tag{3.11}$$

for all $t \geq 0$, where $\{q(t), \Sigma(t) | t \geq 0\}$ is some two-dimensional dynamical system where the $\Sigma(t)$ process is always positive definite. We call this second order approximation the *Gaussian Variance Approximation*. Now if we substitute our approximate distribution of the queueing process, Eq. 3.11, into the functional forward equations we obtain our second approximation and second theorem.

Theorem 3.2 *If we assume that the surrogate distribution for the queueing processes follows a bivariate normal distribution i.e.*

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \approx \begin{pmatrix} q_1 + \sqrt{v_1} \cdot X \\ q_2 + \sqrt{v_2} \cdot (X \cdot \cos\theta + Y \cdot \sin\theta) \end{pmatrix}$$

where X, Y are independent standard Gaussian random variables, then we have

$$\begin{aligned}
\overset{\bullet}{q}_1 &= \lambda_1(t) - \mu_1 \cdot E\left[\{X > \chi_1\} \cdot \{X \cdot \cos\theta \right. \\
&\quad \left. + Y \cdot \sin\theta > \chi_2\}\right] \\
&\quad - (\mu_1 + \mu_2) \cdot E\left[\{X > \chi_1\} \cdot \{X \cdot \cos\theta \right. \\
&\quad \left. + Y \cdot \sin\theta \leq \chi_2\}\right] - \beta_1 \cdot q_1 \\
\overset{\bullet}{q}_2 &= \mu_1 \cdot E\left[\{X > \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}\right] \\
&\quad + (\mu_1 + \mu_2) \cdot E\left[\{X > \chi_1\} \cdot \{X \cdot \cos\theta \right. \\
&\quad \left. + Y \cdot \sin\theta \leq \chi_2\}\right] \\
&\quad - \mu_2 \cdot E\left[\{X > \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}\right] \\
&\quad - (\mu_1 + \mu_2) \cdot E\left[\{X \leq \chi_1\} \cdot \{X \cdot \cos\theta \right. \\
&\quad \left. + Y \cdot \sin\theta > \chi_2\}\right] - \beta_2 \cdot E[Q_2]
\end{aligned}$$

for the mean queue length and

$$\begin{aligned}
\overset{\bullet}{v}_1 &= \lambda_1(t) + \mu_1 \cdot E\left[\{X > \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}\right] \\
&\quad + (\mu_1 + \mu_2) \cdot E\left[\{X > \chi_1\} \cdot \{X \cdot \cos\theta \right. \\
&\quad \left. + Y \cdot \sin\theta \leq \chi_2\}\right] + \beta_1 \cdot E[Q_1] \\
&\quad - 2 \cdot \mu_1 \cdot \sqrt{v_1} \cdot \mathrm{Cov}\left[X, \{X > \chi_1\} \cdot \{X \cdot \cos\theta \right. \\
&\quad \left. + Y \cdot \sin\theta > \chi_2\}\right]
\end{aligned}$$

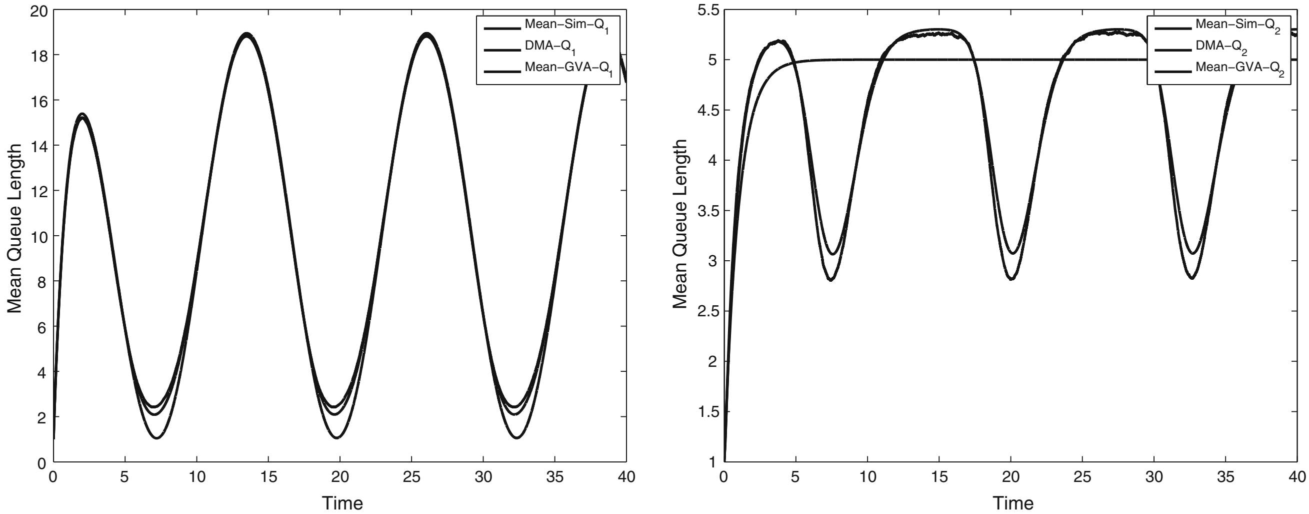

Fig. 4 Comparison of simulated, DMA, and GVA means of Q_1 (*Left*). Comparison of Simulated, DMA, and GVA Means of Q_2 (*Right*)

$$
\begin{aligned}
&- 2 \cdot (\mu_1 + \mu_2) \cdot \sqrt{v_1} \cdot \mathrm{Cov}\,[X, \{X > \chi_1\} \\
&\cdot \{X \cdot \cos\theta + Y \cdot \sin\theta \le \chi_2\}] \\
&- 2 \cdot \beta_1 \cdot v_1 \\
\dot{v}_2 = {}& \mu_1 \cdot E\,[\{X > \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\
&+ \mu_1 + \mu_2) \cdot E\,[\{X > \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta \le \chi_2\}] \\
&+ \mu_2 \cdot E\,[\{X > \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\
&+ (\mu_1 + \mu_2) \cdot E\,[\{X \le \chi_1\} \cdot \{X \cdot \cos\theta \\
&+ Y \cdot \sin\theta > \chi_2\}] + \beta_2 \cdot q_2 \\
&- 2 \cdot \mu_2 \cdot \sqrt{v_2} \cdot \mathrm{Cov}\,[X \cdot \cos\theta + Y \cdot \sin\theta, \{X > \chi_1\} \\
&\cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\
&- 2 \cdot (\mu_1 + \mu_2) \cdot \sqrt{v_2} \cdot \mathrm{Cov}\,[X \cdot \cos\theta + Y \\
&\cdot\ \sin\theta, \{X \le \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\
&- 2 \cdot \beta_2 \cdot v_2 \\
\dot{v}_3 = {}& -\mu_1 \cdot \sqrt{v_2} \cdot \mathrm{Cov}\,[X \cdot \cos\theta + Y \cdot \sin\theta, \{X > \chi_1\} \\
&\cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\
&- \mu_2 \cdot \sqrt{v_1} \cdot \mathrm{Cov}\,[X, \{X > \chi_1\} \cdot \{X \cdot \cos\theta \\
&+ Y \cdot \sin\theta > \chi_2\}] \\
&- (\mu_1 + \mu_2) \cdot \sqrt{v_2} \cdot \mathrm{Cov}\,[X \cdot \cos\theta + Y \\
&\cdot\ \sin\theta, \{X > \chi_1\} \cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\
&- (\mu_1 + \mu_2) \cdot \sqrt{v_1} \cdot \mathrm{Cov}\,[X, \{X \le \chi_1\} \cdot \{X \\
&\cdot\ \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\
&- (\beta_1 + \beta_2) \cdot v_3
\end{aligned}
$$

for the variance and covariance of the queue length process, where $\chi_1 = \frac{-q_1}{\sqrt{v_1}}$ *and* $\chi_2 = \frac{-q_2}{\sqrt{v_2}}$.

Proof In order to compute the mean, variance, and covariance of the queue length processes, then we need to compute closed form expressions for the following expectation and covariance terms

$$
\begin{aligned}
&E[\{Q_1 > 0\}] \\
&E[\{Q_2 > 0\}] \\
&E[\{Q_1 > 0\} \cdot \{Q_2 > 0\}] \\
&E[\{Q_1 > 0\} \cdot \{Q_2 \le 0\}] \\
&E[\{Q_1 \le 0\} \cdot \{Q_2 > 0\}] \\
&\mathrm{Cov}\,[Q_1, \{Q_1 > 0\} \cdot \{Q_2 > 0\}] \\
&\mathrm{Cov}\,[Q_1, \{Q_1 > 0\} \cdot \{Q_2 \le 0\}] \\
&\mathrm{Cov}\,[Q_1, \{Q_1 \le 0\} \cdot \{Q_2 > 0\}] \\
&\mathrm{Cov}\,[Q_2, \{Q_1 > 0\} \cdot \{Q_2 > 0\}] \\
&\mathrm{Cov}\,[Q_2, \{Q_1 > 0\} \cdot \{Q_2 \le 0\}] \\
&\mathrm{Cov}\,[Q_2, \{Q_1 \le 0\} \cdot \{Q_2 > 0\}]
\end{aligned}
$$

We compute these expectations and covariance terms in the Appendix. □

Remark 3.3 We should mention that in our representation of the queue length in terms of trigonometric functions like

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \approx \begin{pmatrix} q_1 + \sqrt{v_1} \cdot X \\ q_2 + \sqrt{v_2} \cdot (X \cdot \cos\theta + Y \cdot \sin\theta) \end{pmatrix}$$

should not confuse readers as it can be easily represented as

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \approx \begin{pmatrix} q_1 + \sqrt{v_1} \cdot X \\ q_2 + \sqrt{v_2} \cdot \left(X \cdot \rho + Y \cdot \sqrt{1 - \rho^2}\right) \end{pmatrix}$$

where $\rho = \cos\theta$ and it is understood that the correlation between the queues is ρ.

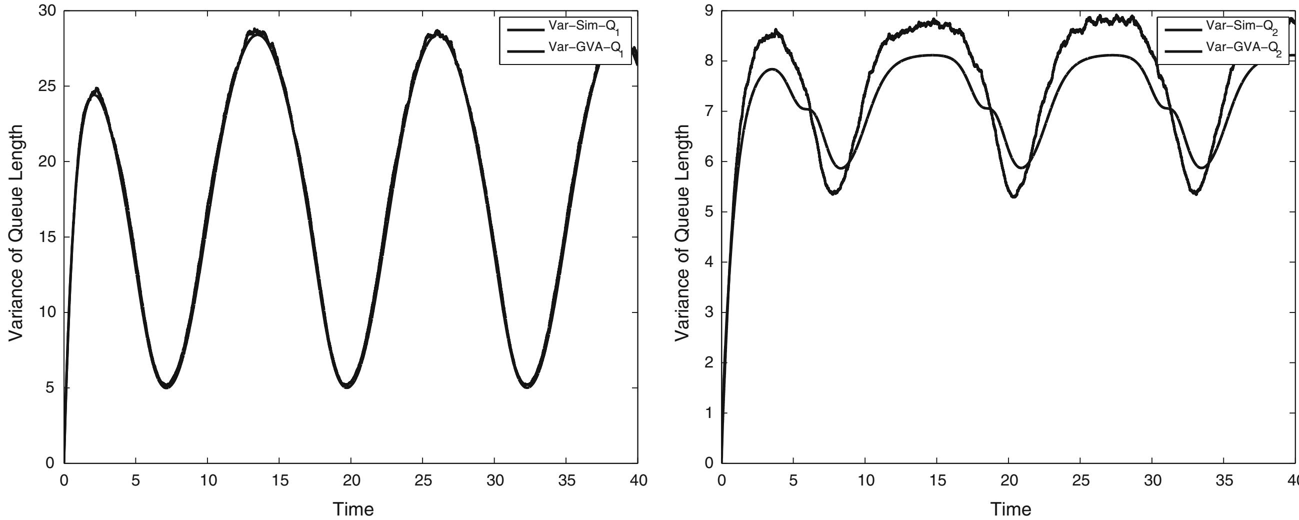

Fig. 5 Comparison of simulated and GVA variances of Q_1 (*Left*). Comparison of simulated and GVA variances of Q_2 (*Right*)

On the left of Fig. 4, we compare the approximations from DMA and GVA for the mean queue length of the first queue. We see that the GVA approximation does the best at estimating the mean dynamics of the first queueing process. This is also true on the right of Fig. 4 where the approximations of DMA and GVA are displayed. We also see that GVA is doing a better job of reproducing the mean dynamics of the simulated behavior. In fact, on the right of Fig. 4 we see that the variance helps estimate the true dynamics of the mean queue length since it includes information about the probabilistic nature of the stochastic queue length process. On the left of Fig. 5, we see that the GVA is doing a good job of approximating the variance of the queue length of the first station. Moreover, we see that the GVA also does a good job of approximating the variance of the queue length of the second station. Lastly, we see in Fig. 6 that the GVA does a good job of estimating the covariance of the two queueing stations. Similar to DMA, we can view GVA as a five dimensional projection of the two stations.

3.4 Probability of emptiness

In addition to approximating the mean and variance of the queue length processes, we can also derive an approximation for the probability that each queue is empty.

Proposition 3.4 *Under the assumption of the GVA, we have the following approximation for the probability that each queue is empty*

$$\mathbb{P}(Q_1 = 0) = 1 - \mathbb{P}(Q_1 > 0) = \overline{\Phi}(\chi_1) \quad (3.12)$$

$$\mathbb{P}(Q_2 = 0) = 1 - \mathbb{P}(Q_2 > 0) = \overline{\Phi}(\chi_2) \quad (3.13)$$

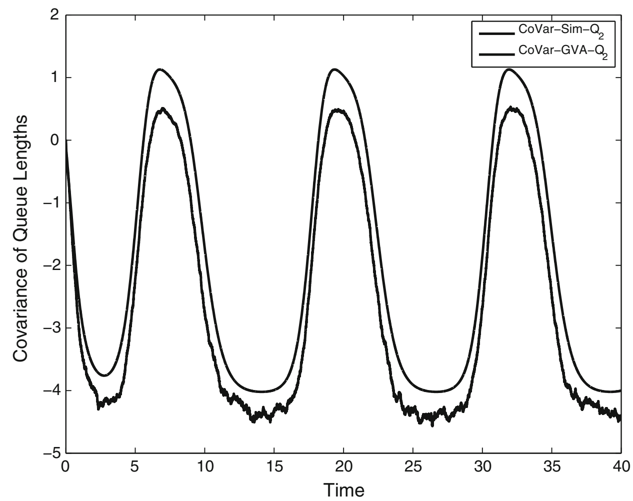

Fig. 6 Comparison of simulated and GVA covariances of Q_1 and Q_2

Proof

$$\begin{aligned}\mathbb{P}(Q_1 = 0) &= 1 - E[\{Q_1 > 0\}] \\ &= E[\{X > \chi_1\}] \\ &= \overline{\Phi}(\chi_1) \\ \mathbb{P}(Q_2 = 0) &= 1 - E[\{Q_2 > 0\}] \\ &= E[\{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] \\ &= \overline{\Phi}(\chi_2)\end{aligned}$$

□

3.5 Additional numerical example

To get a better understanding of the coupled processor, we simulate the queueing process. For the example that we will consider throughout the paper, we assume that the arrival rate is $20 + 10 \cdot \sin(t)$, the service rate for the first queue is

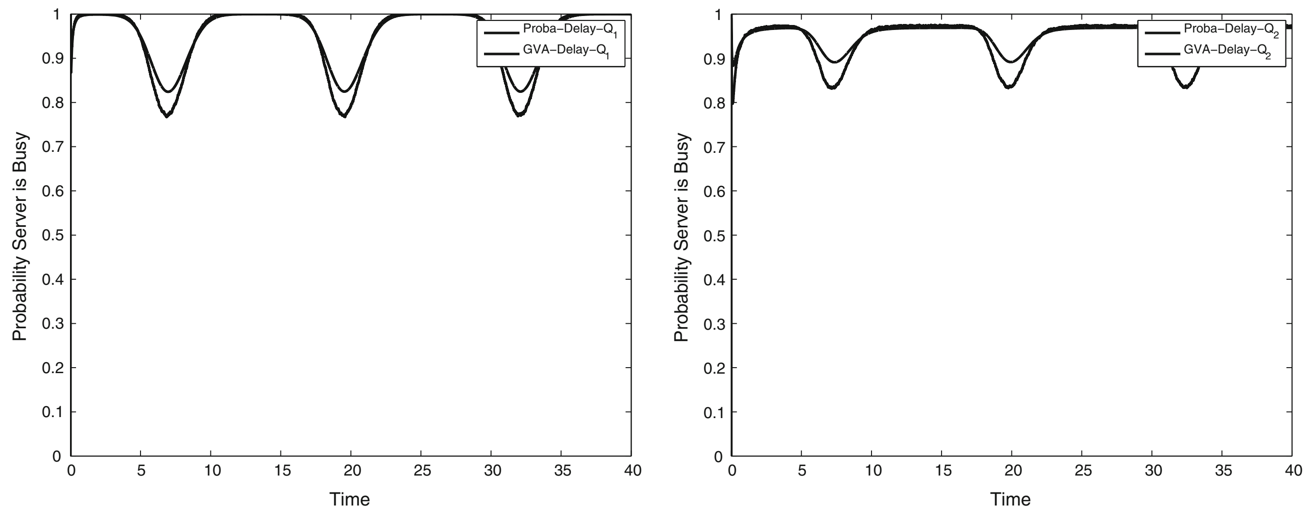

Fig. 7 Simulated probability of emptiness vs. GVA of queue 1 (*Left*). Simulated probability of emptiness vs. GVA of queue 2 (*Right*)

$\mu_1 = 10$, the service rate for the second queue is $\mu_2 = 5$, and the abandonment rates for each queue is $\beta_1 = \beta_2 = 1$. We also simulate the system over the time interval of (0,40], with a time step of $\Delta t = .001$ for 10,000 sample paths.

4 Conclusions and future work

We propose a new method for approximating the time varying dynamics of a coupled processor tandem queueing system. Unlike much of the literature, we consider the model with a time varying arrival rate and the possibility of abandonment. Our method is not only useful for computing accurate estimates of the mean and variance, but also the probability that each queue is empty. To derive these approximations we give closed form approximations for the calculation of two dimensional Gaussian integrals.

We should also mention that unlike generating functions in the two dimensional case, our method extends beyond the two dimensional example we propose in the paper. Using our method, we can extend our results to the case of a n-dimensional coupled processor and also when the arrival processes are of batch type. The functional forward equations for this n-dimensional setting are

$$\begin{aligned}\overset{\bullet}{E}[f(Q)] &= \sum_{j=1}^{n} \lambda_j(t) \cdot E\left[f(Q+e_j) - f(Q)\right] \\ &\quad + \sum_{j=1}^{2^n} E\left[\left(f(Q-g_j(e)) - f(Q)\right) \cdot P_j(Q)\right] \\ &\quad + \sum_{j=1}^{n} \beta_j \cdot E\left[\left(f(Q-e_j) - f(Q)\right) \cdot Q_j\right]\end{aligned}$$

where $P_j(Q)$ represents the partition function that divides up the service rates of all the queues that are non-zero. All of the complexity is hidden in the function $P_j(Q)$ since it incorporates when the queueing processes are empty and at what rate each queue should be served at.

However, this would require the computation of n-dimensional Gaussian integrals, which are more difficult and we do not consider this extension here. Perhaps an independence assumption like in the propagation of chaos or the work of Pender [23] might yield some simple approximations for the n-dimensional case. Another potential method that could be used is the sampling method of Pender [20–24]. Moreover, our approach can be extended to the model of coupled processors considered by Knessl and Morrison [10] where the two queues are fed from two different arrival streams and work in parallel instead of in a tandem fashion. This extension, like the priority extension of Johan et al. case requires no new calculations since the forward equations are quite similar.

Lastly, we intend to apply our polynomial chaos technique to the work of Boxma et al.[5] where they study the workload process with non-exponential service distributions. This is actually more realistic since the workload process is not a discrete process and has a continuous density since it is a spectrally positive Levy process. This case also might benefit from methods in Pender [17,20–23] as well (Fig. 7).

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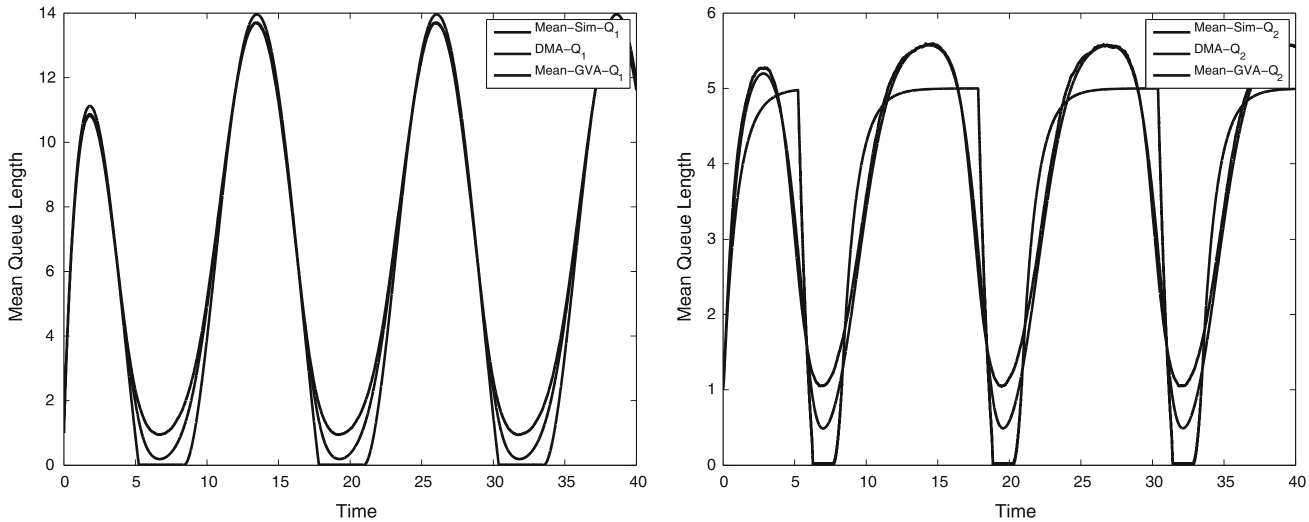


Fig. 8 Simulated mean vs. GVA mean of queue 1 (*Left*). Simulated mean vs. GVA mean of queue 2 (*Right*)

5 Appendix

5.1 Hermite polynomials

Lemma 5.1 (Stein [26]). *The random variable X is Gaussian $(0,1)$ if and only if*

$$E\left[X \cdot f(X)\right] = E\left[\frac{d}{dX} f(X)\right], \tag{5.1}$$

for all generalized functions f. Moreover, we also have that

$$E\left[h_n(X) \cdot f(X)\right] = E\left[\frac{d^n}{dX^n} f(X)\right], \tag{5.2}$$

where $h_n(X)$ is the n^{th} Hermite polynomial.

Let X and Y be two i.i.d Gaussian(0,1) random variables (Fig. 8).

Proposition 5.2 *Any L^2 function can be written as an infinite sum of Hermite polynomials of X, i.e.*

$$f(X,Y) \overset{L^2}{=} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!} E\left[\frac{\partial^{n+m} f}{\partial^m X \partial^n Y}(X,Y)\right] \cdot h_m(X) \cdot h_n(Y),$$

$$E[f(X,Y) \cdot g(X,Y)] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!} E\left[\frac{\partial^{n+m} f}{\partial^m X \partial^n Y}(X,Y)\right] \cdot E\left[\frac{\partial^{n+m} g}{\partial^m X \partial^n Y}(X,Y)\right]$$

and

$$\mathrm{Cov}[f(X), g(X,Y)] = \sum_{m=1}^{\infty} \frac{1}{m!} E\left[\frac{\partial^m f}{\partial^m X}(X)\right] \cdot E\left[\frac{\partial^m g}{\partial^m X}(X,Y)\right]$$

5.2 Calculation of expectation and covariance terms

We define φ and Φ to be the density and the cumulative distribution functions, for X respectively, where

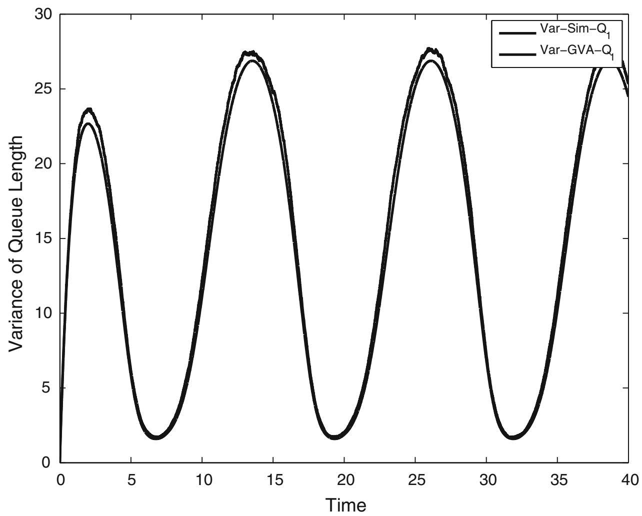
$$\varphi(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \Phi(x) \equiv \int_{-\infty}^{x} \varphi(y)\,dy, \quad \text{and}$$
$$\overline{\Phi}(x) \equiv 1 - \Phi(x) = \int_{x}^{\infty} \varphi(y)\,dy. \tag{5.3}$$

We begin with some of the simpler expectation terms that only involve the evaluation of the Gaussian tail cdf.

$$E[\{X > \chi_1\}] = \mathbb{P}(X > \chi_1) = \overline{\Phi}(\chi_1)$$
$$E[\{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}] = \mathbb{P}(Z > \chi_2) = \overline{\Phi}(\chi_2)$$

Now we use the previous proposition to derive the following expectations. Using the L^2 expansion of the function, we get an infinite series representation for the first line. To move from the second to the third line, we use the fact that the function $\{Q_1 > 0\}$ does not depend on the function Y. Lastly, we use the Hermite polynomial generalization of Stein's lemma (Fig. 9).

$$\mathrm{E}\left[\{Q_1 > 0\} \cdot \{Q_2 > 0\}\right]$$



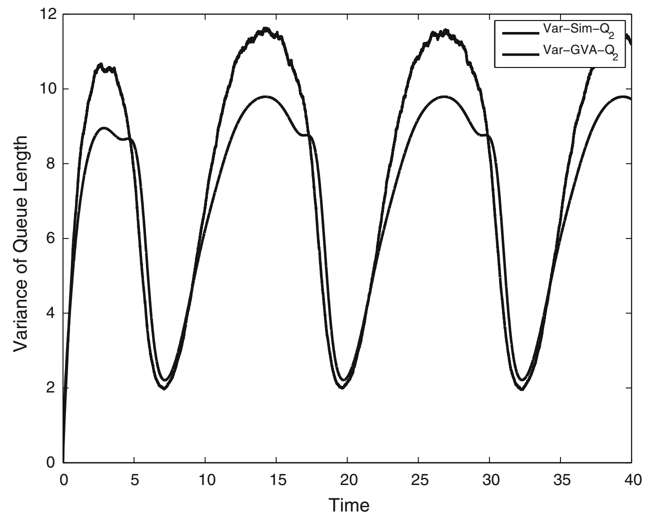


Fig. 9 Simulated variance vs. GVA variance of queue 1 (*Left*). Simulated variance vs. GVA variance of queue 2 (*Right*)

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}E\left[\frac{\partial^{n+m}}{\partial^m X\partial^n Y}\{Q_1>0\}\right]\\
&\quad\cdot E\left[\frac{\partial^{n+m}}{\partial^m X\partial^n Y}\{Q_2>0\}\right]\\
&= \sum_{m=0}^{\infty}\frac{1}{m!}E\left[\frac{\partial^m}{\partial^m X}\{Q_1>0\}\right]\cdot E\left[\frac{\partial^m}{\partial^m X}\{Q_2>0\}\right]\\
&\quad\text{(since } Q_1 \text{ does not depend on Y)}\\
&= \overline{\Phi}(\chi_1)\cdot\overline{\Phi}(\chi_2)+\phi(\chi_1)\cdot\phi(\chi_2)\cdot\sum_{m=1}^{\infty}\frac{1}{m!}\\
&\quad\cdot h_{m-1}(\chi_1)\cdot h_{m-1}(\chi_2)\cdot\cos^m\theta.
\end{aligned}
$$

The following two expectations can be calculated easily using the previous calculations.

$$
\begin{aligned}
&\mathrm{E}\left[\{Q_1>0\}\cdot\{Q_2\le 0\}\right]\\
&\quad=\mathrm{E}\left[\{Q_1>0\}\right]-\mathrm{E}\left[\{Q_1>0\}\cdot\{Q_2>0\}\right]\\
&\quad=\overline{\Phi}(\chi_1)-\overline{\Phi}(\chi_1)\cdot\overline{\Phi}(\chi_2)-\phi(\chi_1)\cdot\phi(\chi_2)\\
&\qquad\cdot\sum_{m=1}^{\infty}\frac{1}{m!}\cdot h_{m-1}(\chi_1)\cdot h_{m-1}(\chi_2)\cdot\cos^m\theta\\
&\mathrm{E}\left[\{Q_1\le 0\}\cdot\{Q_2>0\}\right]\\
&\quad=\mathrm{E}\left[\{Q_2>0\}\right]-\mathrm{E}\left[\{Q_1>0\}\cdot\{Q_2>0\}\right]\\
&\quad=\overline{\Phi}(\chi_2)-\overline{\Phi}(\chi_1)\cdot\overline{\Phi}(\chi_2)-\phi(\chi_1)\cdot\phi(\chi_2)\\
&\qquad\cdot\sum_{m=1}^{\infty}\frac{1}{m!}\cdot h_{m-1}(\chi_1)\cdot h_{m-1}(\chi_2)\cdot\cos^m\theta
\end{aligned}
$$

Now we begin the calculation of the covariance terms with respect to the first queue length. From the first line to the second we use the property that covariances are invariant to constants. Then, we use the Hermite polynomial expansion property and the Hermite polynomial generalization of Stein's lemma once again (Fig. 10).

$$
\begin{aligned}
&\mathrm{Cov}\left[Q_1,\{Q_1>0\}\cdot\{Q_2>0\}\right]\\
&\quad=\mathrm{Cov}\left[q_1+\sqrt{v_1}\cdot X,\{Q_1>0\}\cdot\{Q_2>0\}\right]\\
&\quad=\sqrt{v_1}\cdot\mathrm{Cov}\left[X,\{Q_1>0\}\cdot\{Q_2>0\}\right]\\
&\quad=\sqrt{v_1}\cdot\mathrm{E}\left[X\cdot\{Q_1>0\}\cdot\{Q_2>0\}\right]\\
&\quad=\sqrt{v_1}\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}E\left[\frac{\partial^{n+m}}{\partial^m X\partial^n Y}X\cdot\{Q_1>0\}\right]\\
&\qquad\cdot E\left[\frac{\partial^{n+m}}{\partial^m X\partial^n Y}\{Q_2>0\}\right]\\
&\quad=\sqrt{v_1}\cdot\sum_{m=0}^{\infty}\frac{1}{m!}E\left[\frac{\partial^m}{\partial^m X}X\cdot\{Q_1>0\}\right]\\
&\qquad\cdot E\left[\frac{\partial^m}{\partial^m X}\{Q_2>0\}\right]\\
&\quad=\sqrt{v_1}\cdot\sum_{m=0}^{\infty}\frac{1}{m!}E\left[\frac{\partial^m}{\partial^m X}X\cdot\{X>\chi_1\}\right]\\
&\qquad\cdot E\left[\frac{\partial^m}{\partial^m X}\{X\cdot\cos\theta+Y\cdot\sin\theta>\chi_2\}\right]\\
&\quad=\sqrt{v_1}\cdot\sum_{m=0}^{\infty}\frac{1}{m!}E\left[h_m(X)\cdot X\cdot\{X>\chi_1\}\right]\\
&\qquad\cdot E\left[\frac{\partial^m}{\partial^m X}\{X\cdot\cos\theta+Y\cdot\sin\theta>\chi_2\}\right]\\
&\quad=\sqrt{v_1}\cdot\sum_{m=0}^{\infty}\frac{1}{m!}E\left[(h_{m+1}(X)+m\cdot h_{m-1}(X))\right.\\
&\qquad\left.\cdot\{X>\chi_1\}\right]\cdot E\left[\frac{\partial^m}{\partial^m X}\{X\cdot\cos\theta+Y\cdot\sin\theta>\chi_2\}\right]\\
&\quad=\sqrt{v_1}\cdot\varphi(\chi_1)\cdot\overline{\Phi}(\chi_2)+\sqrt{v_1}\cdot\left(\overline{\Phi}(\chi_1)\right.
\end{aligned}
$$

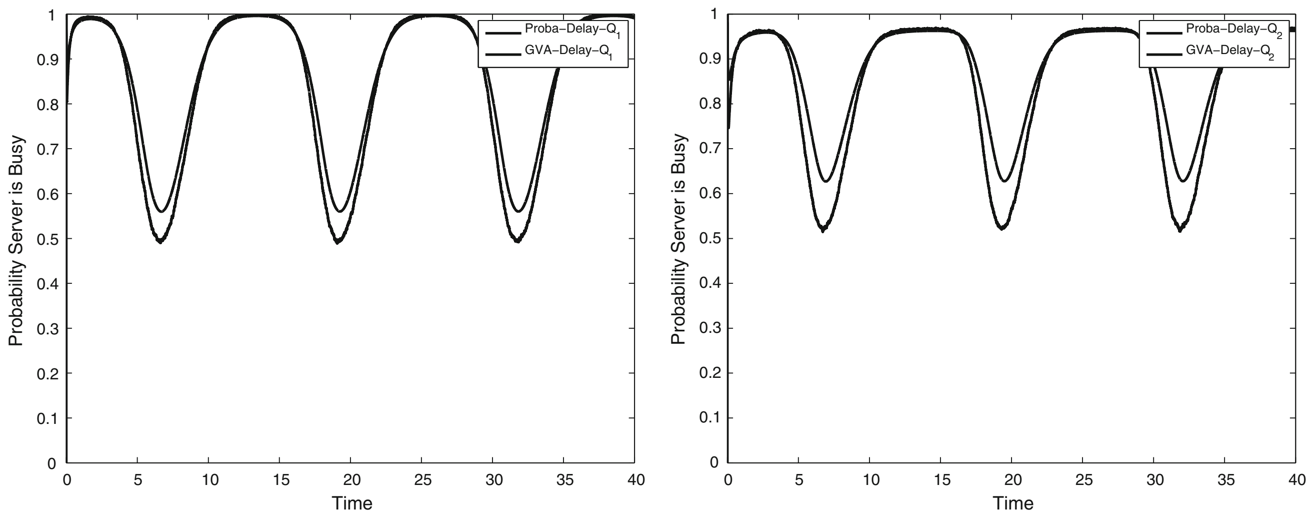


Fig. 10 Simulated probability of emptiness of queue 1 (*Left*). Simulated probability of emptiness of queue 2 (*Right*)

$$
\begin{aligned}
&+ \chi_1 \cdot \varphi(\chi_1)) \cdot \varphi(\chi_2) \cdot \cos\theta \\
&+ \sqrt{v_1} \cdot \sum_{m=2}^{\infty} \frac{1}{m!} \left((h_m(\chi_1) + m \cdot h_{m-2}(\chi_1)) \cdot \varphi(\chi_1) \cdot\right) \\
&\cdot E\left[\frac{\partial^m}{\partial^m X}\{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}\right] \\
=\ & \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \overline{\Phi}(\chi_2) + \sqrt{v_1} \cdot \big(\overline{\Phi}(\chi_1) + \chi_1 \\
&\cdot \varphi(\chi_1)) \cdot \varphi(\chi_2) \cdot \cos\theta \\
&+ \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \varphi(\chi_2) \cdot \sum_{m=2}^{\infty} \frac{1}{m!} (h_m(\chi_1) \\
&+ m \cdot h_{m-2}(\chi_1)) \cdot h_{m-1}(\chi_2) \cdot \cos^m\theta
\end{aligned}
$$

For the next two covariance terms, we use the previous covariance term in the calculation.

$$
\begin{aligned}
&\mathrm{Cov}\left[Q_1, \{Q_1 > 0\} \cdot \{Q_2 \le 0\}\right] \\
&= \mathrm{Cov}\left[Q_1, \{Q_1 > 0\} \cdot (1 - \{Q_2 > 0\})\right] \\
&= \mathrm{Cov}\left[Q_1, \{Q_1 > 0\}\right] - \mathrm{Cov}\left[Q_1, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\
&= \mathrm{Cov}\left[\sqrt{v_1} \cdot X, \{X > \chi_1\}\right] - \mathrm{Cov}\left[\sqrt{v_1} \cdot X, \{X > \chi_1\}\right. \\
&\quad \left.\cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}\right] \\
&= \sqrt{v_1} \cdot \varphi(\chi_1) - \sqrt{v_1} \cdot \mathrm{Cov}\left[X, \{X > \chi_1\}\right. \\
&\quad \left.\cdot \{X \cdot \cos\theta + Y \cdot \sin\theta > \chi_2\}\right] \\
&= \sqrt{v_1} \cdot \varphi(\chi_1) - \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \overline{\Phi}(\chi_2) \\
&\quad - \sqrt{v_1} \cdot \left(\overline{\Phi}(\chi_1) + \chi_1 \cdot \varphi(\chi_1)\right) \cdot \varphi(\chi_2) \cdot \cos\theta \\
&\quad - \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \varphi(\chi_2) \cdot \sum_{m=2}^{\infty} \frac{1}{m!} (h_m(\chi_1) \\
&\quad + m \cdot h_{m-2}(\chi_1)) \cdot h_{m-1}(\chi_2) \cdot \cos^m\theta \\
&= \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \Phi(\chi_2) - \sqrt{v_1} \cdot \left(\overline{\Phi}(\chi_1) + \chi_1 \cdot \varphi(\chi_1)\right) \\
&\quad \cdot \varphi(\chi_2) \cdot \cos\theta \\
&\quad - \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \varphi(\chi_2) \cdot \sum_{m=2}^{\infty} \frac{1}{m!} (h_m(\chi_1) \\
&\quad + m \cdot h_{m-2}(\chi_1)) \cdot h_{m-1}(\chi_2) \cdot \cos^m\theta
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Cov}\left[Q_1, \{Q_1 \le 0\} \cdot \{Q_2 > 0\}\right] \\
&= \mathrm{Cov}\left[Q_1, (1 - \{Q_1 > 0\}) \cdot \{Q_2 > 0\}\right] \\
&= \mathrm{Cov}\left[Q_1, \{Q_2 > 0\}\right] - \mathrm{Cov}\left[Q_1, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\
&= \mathrm{Cov}\left[\sqrt{v_1} \cdot X, \{Q_2 > 0\}\right] - \mathrm{Cov}\left[\sqrt{v_1} \cdot X, \{Q_1 > 0\}\right. \\
&\quad \left.\cdot \{Q_2 > 0\}\right] \\
&= \sqrt{v_1} \cdot \varphi(\chi_2) \cdot \cos\theta - \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \overline{\Phi}(\chi_2) \\
&\quad - \sqrt{v_1} \cdot \left(\overline{\Phi}(\chi_1) + \chi_1 \cdot \varphi(\chi_1)\right) \cdot \varphi(\chi_2) \cdot \cos\theta \\
&\quad - \sqrt{v_1} \cdot \varphi(\chi_1) \cdot \varphi(\chi_2) \cdot \sum_{m=2}^{\infty} \frac{1}{m!} (h_m(\chi_1) \\
&\quad + m \cdot h_{m-2}(\chi_1)) \cdot h_{m-1}(\chi_2) \cdot \cos^m\theta
\end{aligned}
$$

Now we begin the calculation of the covariance terms with respect to the second queue length. From the first line to the second we use the property that covariances are invariant to constants. Then, we use the Hermite polynomial expansion property and the Hermite polynomial generalization of Stein's lemma once again.

$$
\begin{aligned}
&\mathrm{Cov}\left[Q_2, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\
&= \mathrm{Cov}\left[\sqrt{v_1} \cdot X, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\
&= \sqrt{v_2} \cdot \mathrm{Cov}\left[X \cdot \cos\theta + Y \cdot \sin\theta, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\
&= \sqrt{v_2} \cdot \cos\theta \cdot \mathrm{Cov}\left[X, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\
&\quad + \sqrt{v_2} \cdot \sin\theta \cdot \mathrm{Cov}\left[Y, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] \\
&= \sqrt{v_2} \cdot \cos\theta \cdot \mathrm{Cov}\left[X, \{Q_1 > 0\} \cdot \{Q_2 > 0\}\right] + \sqrt{v_2} \\
&\quad \cdot \sin\theta \cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} E\left[\frac{\partial^{n+m}}{\partial^m X \partial^n Y} Y \cdot \{Q_1 > 0\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\cdot E\left[\frac{\partial^{n+m}}{\partial^m X \partial^n Y}\{Q_2>0\}\right] \\
&= \sqrt{v_2}\cdot\cos\theta\cdot \mathrm{Cov}\left[X,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\quad +\sqrt{v_2}\cdot\sin\theta\cdot\sum_{m=0}^{\infty}\frac{1}{m!}E\left[\frac{\partial^{1+m}}{\partial^m X\partial Y}Y\cdot\{Q_1>0\}\right] \\
&\quad \cdot E\left[\frac{\partial^{1+m}}{\partial^m X\partial Y}\{Q_2>0\}\right] \\
&= \sqrt{v_2}\cdot\cos\theta\cdot \mathrm{Cov}\left[X,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\quad +\sqrt{v_2}\cdot\sin\theta\cdot\left(\Phi(\chi_1)\cdot\varphi(\chi_2)\cdot\sin\theta\right)+\sqrt{v_2} \\
&\quad \cdot\sin\theta\cdot(\varphi(\chi_1)\cdot\chi_2\cdot\varphi(\chi_2)\cdot\sin\theta\cdot\cos\theta) \\
&\quad +\sqrt{v_2}\cdot\sin\theta\cdot\sum_{m=2}^{\infty}\frac{1}{m!}E\left[\frac{\partial^m}{\partial^m X}\{X>\chi_1\}\right] \\
&\quad \cdot E\left[\frac{\partial^m}{\partial^m X}\delta_{\chi_2}(X\cdot\cos\theta+Y\cdot\sin\theta)\right] \\
&= \sqrt{v_2}\cdot\cos\theta\cdot \mathrm{Cov}\left[X,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\quad +\sqrt{v_2}\cdot\sin\theta\cdot\left(\Phi(\chi_1)\cdot\varphi(\chi_2)\cdot\sin\theta\right)+\sqrt{v_2}\cdot\sin\theta \\
&\quad \cdot(\varphi(\chi_1)\cdot\chi_2\cdot\varphi(\chi_2)\cdot\sin\theta\cdot\cos\theta) \\
&\quad +\sqrt{v_2}\cdot\sin\theta\cdot\sum_{m=2}^{\infty}\frac{1}{m!}\varphi(\chi_1)\cdot h_{m-1}(\chi_1)\cdot\varphi(\chi_2) \\
&\quad \cdot h_m(\chi_2)\cdot\sin\theta\cdot\cos^m\theta
\end{aligned}
$$

Lastly, for the next two covariance terms, we use the previous covariance term in the calculation.

$$
\begin{aligned}
&\mathrm{Cov}\left[Q_2,\{Q_1>0\}\cdot\{Q_2\le 0\}\right] \\
&\quad = \mathrm{Cov}\left[Q_2,\{Q_1>0\}\cdot(1-\{Q_2>0\})\right] \\
&\quad = \mathrm{Cov}\left[Q_2,\{Q_1>0\}\right]-\mathrm{Cov}\left[Q_2,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\quad = \sqrt{v_2}\cdot\mathrm{Cov}\left[X\cdot\cos\theta+Y\cdot\sin\theta,\{Q_1>0\}\right]-\sqrt{v_2} \\
&\qquad \cdot\mathrm{Cov}\left[X\cdot\cos\theta+Y\cdot\sin\theta,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\quad = \sqrt{v_2}\cdot\cos\theta\cdot\varphi(\chi_1)-\sqrt{v_2}\cdot\mathrm{Cov}\left[X\cdot\cos\theta\right. \\
&\qquad \left.+Y\cdot\sin\theta,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\mathrm{Cov}\left[Q_2,\{Q_1\le 0\}\cdot\{Q_2>0\}\right] \\
&\quad = \mathrm{Cov}\left[Q_2,(1-\{Q_1>0\})\cdot\{Q_2>0\}\right] \\
&\quad = \mathrm{Cov}\left[Q_2,\{Q_2>0\}\right]-\mathrm{Cov}\left[Q_2,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\quad = \sqrt{v_2}\cdot\mathrm{Cov}\left[X\cdot\cos\theta+Y\cdot\sin\theta,\{Q_2>0\}\right]-\sqrt{v_2} \\
&\qquad \cdot\mathrm{Cov}\left[X\cdot\cos\theta+Y\cdot\sin\theta,\{Q_1>0\}\cdot\{Q_2>0\}\right] \\
&\quad = \sqrt{v_2}\cdot\varphi(\chi_2)-\sqrt{v_2}\cdot\ \mathrm{Cov}\left[X\cdot\cos\theta+Y\right. \\
&\qquad \left.\cdot\sin\theta,\{Q_1>0\}\cdot\{Q_2>0\}\right]
\end{aligned}
$$

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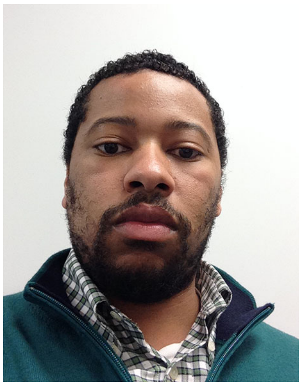

Jamol Pender received the Ph. D degree from the department of Operations Research and Financial Engineering at Princeton University in 2013. He also received a BSE and MSE in Electrical and Systems Engineering from the University of Pennsylvania in 2008. He is currently an Assistant Professor in the School of Operations Research and Information Engineering at Cornell University. His main research interests are applied probability, nonstationary queueing networks, performance analysis of queues, Markov processes, and unobservable queues.