INTEGRATED-QUANTILE-BASED ESTIMATION FOR FIRST-PRICE AUCTION MODELS

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ABSTRACT. This paper considers nonparametric estimation of first-price auction models under the monotonicity restriction on the bidding strategy. Based on an integrated-quantile representation of the first-order condition, we propose a tuning-parameter-free estimator for the valuation quantile function. We establish its cube-root-n consistency and asymptotic distribution under weaker smoothness assumptions than those typically assumed in the empirical literature. If the latter are true, we also provide a trimming-free smoothed estimator and show that it is asymptotically normal and achieves the optimal rate of Guerre, Perrigne, and Vuong (2000). We illustrate our method using Monte Carlo simulations and an empirical study of the California highway procurements auctions.

Key words: First Price Auctions, Monotone Bidding Strategy, Nonparametric Estimation, Tuning-Parameter-Free

JEL: D44, D82, C12, C14

First version: Wednesday 21st January, 2015

This version: Wednesday 15th July, 2015

We thank the valuable comments from Victor Aguirregabiria, Tim Armstrong, Christian Gourieroux, Emmanuel Guerre, Shakeeb Khan, Ruixuan Liu, Vadim Marmer and Ismael Mourifie. We benefited from discussions with participants at CMES 2014, the 2nd CEC and the 5th Shanghai Econometrics Workshop at SUFE. All remaining errors are ours. Luo gratefully acknowledge financial support from the National Natural Science Foundation of China (NSFC-71373283 and NSFC-71463019).

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1. INTRODUCTION

Since the seminal work of Guerre, Perrigne, and Vuong (2000, GPV hereafter), the nonparametric estimation of auction models has received enormous attention from both the perspectives of econometric analysis and empirical applications. In this paper, we revisit the first-price auction models and propose a novel estimation procedure for the valuation quantile function. Our approach is appealing both computationally and theoretically. We first construct a quantile estimator that is tuning-parameter-free and robust in the sense that it is consistent under weaker smoothness assumptions than typically imposed in the literature (details later). Whenever the typical smoothness assumptions are satisfied, we can construct a trimming-free and asymptotically normal second step estimator that achieves the optimal rate of GPV. Furthermore, our estimator explicitly incorporates the restriction of the monotone bidding strategy and is monotone in finite samples, which is important for empirical work but not ensured by most of the existing estimators.

To better illustrate the features of our estimator, we begin by reviewing existing approaches in the literature. We focus on the baseline case of homogeneous auctions and will show it can be extended and incorporate auction specific characteristics in Section 2.3. We consider the standard GPV setup of independent private value (IPV) first price auction. Their novel approach is to transform the first-order condition for optimal bids and express a bidder's value as an explicit function of the submitted bid, the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of bids:

$$v = s^{-1}(b) \equiv b + \frac{1}{I-1}\frac{G(b)}{g(b)}, \tag{1}$$

where b is the bid, I is the number of bidders, and $G(\cdot)$ and $g(\cdot)$ are the distribution and density of bids, respectively. A two-step estimation method follows from this observation: first construct a pseudo value for each bid and then apply kernel density estimation to the sample of pseudo values. GPV establish the consistency of their estimator and the optimal rate.

Based on the insight of Haile, Hong, and Shum (2003), who considered a quantile-based-test for the existence of common values, Marmer and Shneyerov (2012, MS hereafter) first proposed to estimate the valuation distribution based on the quantile representation of the first-order condition, that is, when the equilibrium bidding strategy is strictly monotone, valuation quantile function $Q_v(\cdot)$ can be expressed as

$$Q_v(\alpha) = Q_b(\alpha) + \frac{1}{I-1}\frac{\alpha}{g(Q_b(\alpha))}, \quad 0 \le \alpha \le 1, \tag{2}$$

where $Q_b(\cdot)$ is the bid quantile function. Note that the right-hand side must be strictly increasing in α, too. MS proposed to first estimate $Q_v(\cdot)$ using plug-in estimators for $g(\cdot)$ and $Q_b(\cdot)$, respectively, and subsequently estimate the valuation density using $f(v) = 1/Q_v'(Q_v^{-1}(v))$. MS show that their estimator is asymptotically normal and achieves the optimal rate of GPV. Guerre and Sabbah (2012, GS) observed that the second term on the right hand side of Equation (2) is a known linear function of α multiplied by the quantile derivative and proposed an optimal local polynomial quantile estimator.

In both estimators of GPV and MS, the bid density $g(\cdot)$ appears in the denominator of the first step estimation; in MS, the derivative of the bid quantile also appears in the denominator of the second step. In practice, trimming near the boundaries is needed but can be troublesome as it is well known that there is no generic guidance. GS does not require trimming but a choice of a bandwidth. In addition, all quantile estimators discussed above may not satisfy the monotonicity restriction imposed by the model.

In this paper, we propose to consider the integrated quantile function of the valuation as in Liu and Luo (2015), who used it for comparing valuation distributions. Define

$$V(\beta) \equiv \int_0^\beta Q_v(\alpha)d\alpha = \frac{I-2}{I-1}\int_0^\beta Q_b(\alpha)d\alpha + \frac{1}{I-1}Q_b(\beta)\beta, \quad 0 \le \beta \le 1. \tag{3}$$

The integrated quantile representation has the following merits. First, the sample analog of $V(\cdot)$, denoted by $V_n(\cdot)$, is easy to compute. It essentially requires little more than sorting the observed bids. Neither bandwidth choice nor trimming is needed. Second, the strict monotonicity of the bidding strategy necessarily implies the strict convexity of the right-hand side. Based on this observation, we can use the greatest convex minorant (g.c.m.) of $V_n(\cdot)$ as an estimator for $V(\cdot)$. We denote the g.c.m. of V_n as $\widehat{V}$. Since $V_n(\cdot)$ is a piece-wise linear function of β, so is $\widehat{V}(\cdot)$, which can be very easily calculated. Then we can estimate $Q_v(\cdot)$ by taking the piece-wise derivatives of $\widehat{V}(\cdot)$. As we will formally prove later, this estimator is cube-root-n consistent and requires weaker smoothness on model primitive, i.e., it only requires that $F(\cdot)$ be continuously differentiable, as opposed to twice continuously differentiable in GPV and MS. We called it as our first step estimator $\widehat{Q}_v(\cdot)$. Note that $\widehat{Q}_v(\cdot)$ is tuning-parameter-free. If indeed the model admits enough smoothness, we can improve the convergence rate by considering a kernel smoothed version $\hat{q}_v(\cdot)$ of $\widehat{Q}_v(\cdot)$. We show that $\hat{q}_v(\cdot)$ is asymptotically normal and achieves GPV's optimal rate. Note that despite that one needs to choose a bandwidth for $\hat{q}_v(\cdot)$ (for which we propose an optimal bandwidth), there is no need for trimming.[1]

Another appealing feature of our estimator is that the monotonicity of bidding strategy is imposed in a simple way through the calculation of g.c.m.. As a result, the estimates $\widehat{Q}_v(\cdot)$ and $\hat{q}_v(\cdot)$ are always increasing by construction. To the best of our knowledge, Henderson, List, Millimet, Parmeter, and Price (2012, HLMPP hereafter) were the first to address the imposition of monotonicity in first price auctions. They argued that nonparametric estimators that naturally impose existing economic restrictions have empirical virtue. Our method, however, is different from theirs, which achieves the desired monotonicity constraint by tilting the empirical distribution of the data by the least amount. Their method requires repeated re-weighting of the sample. Bierens and Song (2012)'s sieve approach implicitly imposes the monotonicity constraint, but it can be computationally expensive. Our estimator imposes the monotonicity by taking the greatest convex minorant of the integrated valuation quantile function. The g.c.m. of $V_n(\cdot)$ is easy to compute since the it is piece-wise linear. As a matter of fact, satisfying monotonicity in finite samples is a desirable feature of a quantile function estimator, see discussions in, for example, Chernozhukov, Fernandez-Val,

[1]See Hickman and Hubbard (2014) for a modified version of the GPV estimator which replaces trimming with boundary correction.

and Galichon (2010). Chernozhukov, Fernandez-Val, and Galichon (2010) proposed a "rearrangement" approach to achieves the monotonicity. We take the g.c.m. approach on the integrate-quantile function in our context because it not only delivers the monotonicity, but also circumvents the necessity of estimating the bid density function in the denominator.

Our estimator is constructed using order statistics of the bids. Indeed, using order statistics is not uncommon in the literature of nonparametric estimation of auction models. See, e.g., Athey and Haile (2007). Recently, Menzel and Morganti (2013) discussed estimation of value distribution based on the distributions of order statistics. They show that the mapping between distribution of order statistics and valuation distribution is in general non-Lipschitz continuous and established optimal rate for varies of parameters of interest. Our main motivation is to provide computationally-easy estimators for the classical IPV setup of GPV with all bids being observed, which is, as mentioned by Menzel and Morganti (2013), a scenario for which the irregularity of inverting order statistics distribution does not rise. The fact that our first estimator converges at an irregular cube-root-n rate is because we impose weaker smoothness assumption on the valuation distribution, rather than the non-Lipschitz continuity of the mapping in other models.

We illustrate our method using the California Highway Procurement auction data set. In practice, it is common that researchers observe auction-specific characteristics.[2] It is worth noting that our method applies naturally if the observed auction-specific characteristics are discrete-valued (or discretization of continuous variables) by conditioning on realizations. The estimate will then be interpreted as conditional valuation quantiles on observed auction characteristics. When the observed auction-specific characteristics are continuous, GPV and MS propose to estimate the conditional valuation density by Kernel method, which suffers the "curse of dimensionality" when the covariates are high dimensional. To overcome such difficulty, Gimenes and Guerre (2013) explored the insight made by GS that the mapping between valuation quantile and bids quantile is linear and proposed an augmented-quantile

[2] In general, the first price auction model is not identified if there is unobserved heterogeneity across auctions, see Armstrong (2013b).

regression method. In Section 2.3, we show that our estimation procedure can also take GS's estimator for bids quantile function as an input and deliver a consistent and monotone conditional valuation quantile estimator for each realization of the continuous covariates. Lastly, we can also use the homogenization method proposed by Haile, Hong, and Shum (2003) and apply our estimation methods to the homogenized bids. The homogenization approach requires additional additive separability structure on how valuation depends on observed characteristics. As a result, it is easier to compute and has faster convergence rate.

The rest of the paper is organized as follows. We lay out the model and propose our estimator in Section 2. We examine the performance of our estimator in Section 3. Section 4 is the empirical illustration. We conclude the paper in Section 5.

2. Model and Main Results

We consider the first-price sealed-bid auction model with independent private values. A single and indivisible object is auctioned. We make the following assumptions.

Assumption 1. *There are $L \to \infty$ identical auctions, and for each auction, there are I symmetric and risk neutral bidders. Their private values are i.i.d. draws from a common distribution $F(\cdot)$.*

Let the total number of bids be $n = LI$. The asymptotics is on the number of auctions, that is, $L \to \infty$. The assumption that number of bidders I is constant across auctions is just for simplifying notation; our analysis can be easily extended to conditional on I.

Assumption 2. *$F(\cdot)$ is continuously differentiable over its compact support $[\underline{v}, \overline{v}]$. There exists $\lambda > 0$ such that $\inf_{v \in [\underline{v}, \overline{v}]} f(v) \geq \lambda > 0$.*

Assumption 2 only requires that $F(\cdot)$ is continuously differentiable, which is weaker than the twice continuously differentiability, as assumed in the literature, e.g., GPV and MS. It is well known that the equilibrium strategy is

$$b = s(v|F, I) \equiv v - \frac{1}{F(v)^{I-1}} \int_0^v F(x)^{I-1} dx.$$

GPV show that the first-order condition can be written as Equation (1). Haile, Hong, and Shum (2003) represents this equation in terms of quantiles as in Equation (2). In this paper, we consider the integrated quantile function of the valuation as in Equation (3).

Now let us first propose a tuning-parameter-free estimator for the valuation quantile function. Let $b_{(i)}$ be the i-th order statistic of a sample of bids $\{b_i\}_{i=1}^n$. Employing Equation (3), we construct a raw estimator $V_n(\cdot)$ for $V(\cdot)$ as follows. Let $V_n(0) = 0$. For $\alpha \in \{\frac{1}{n}, \frac{2}{n}, \cdots, 1\}$,

$$V_n(\alpha) = \frac{I-2}{n(I-1)} \sum_{i=1}^{n\alpha} b_{(i)} + \frac{1}{I-1} \alpha b_{(n\alpha)}.$$

For $\alpha \in \left(\frac{j-1}{n}, \frac{j}{n}\right)$, $j = 1, \cdots, n$, define

$$V_n(\alpha) = (j - \alpha n) V_n\left(\frac{j-1}{n}\right) + (\alpha n - j + 1) V_n\left(\frac{j}{n}\right).$$

Note that $V_n(\cdot)$ may not be convex in finite samples. To obtain a quantile estimator which respects the monotonicity property, we consider use the left-derivative of the g.c.m. of $V_n(\cdot)$. Let $\widehat{V}(\cdot)$ be the g.c.m. of $V_n(\cdot)$. Since $V_n(\cdot)$ is piecewise linear, so is $\widehat{V}(\cdot)$. Define $\widehat{Q}_v(0) = \underline{v}$ and for $\alpha \in \left(\frac{j-1}{n}, \frac{j}{n}\right]$, $j = 1, \cdots, n$,

$$\widehat{Q}_v(\alpha) = n\left\{\widehat{V}\left(\frac{j}{n}\right) - \widehat{V}\left(\frac{j-1}{n}\right)\right\}.$$

By definition, $\widehat{Q}_v(\cdot)$ is a left-continuous and weakly increasing step function.

Constructing the g.c.m. $\widehat{V}(\cdot)$ for a piecewise linear function $V_n(\cdot)$ is computationally easy. While many algorithms are proposed, the most widely used one is the Pooled Adjacent Voilators Algorithm (PAVA, see e.g. Robertson, Wright, Dykstra, and Robertson, 1988; Groeneboom, Jongbloed, and Wellner, 2014). We can envision $\widehat{V}(\cdot)$ as a taut string tied to the left most point $(0,0)$ and pulled up and under the graph of $V_n(\cdot)$, ending at the last point $(1, V_n(1))$. See Appendix D for details.

Theorem 1. *Suppose Assumptions 1 and 2 are satisfied at $\alpha_0 \in (0,1)$, then*

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \xrightarrow{d} C(\alpha_0)\, \mathrm{argmax}_t \left\{ \mathbb{B}(t) - t^2 \right\},$$

where $C(\alpha_0)$ is a constant depends on α_0 and $\mathbb{B}$ is a two-sided Brownian motion process.

Proof. See Appendix A.1. □

We have a few comments on Theorem 1. First, $C(\alpha_0)$ depends on α_0, g and Q_b and is estimable (detailed expression in Appendix A.1). To conduct inference on $Q_v(\alpha_0)$, one can obtain the critical values by estimating $C(\alpha_0)$ and simulating the one-dimensional Brownian motion $\mathbb{B}$, which is easy to compute. An alternative way is subsampling whose validity follows straightforwardly from Theorem 1. Second, the $n^{1/3}$-consistency of our quantile estimator is obtained under weak assumptions on value distribution $F(\cdot)$ and without choosing any tuning parameters. It is slower than the optimal rate of $n^{2/5}$ when $F(\cdot)$ is twice continuously differentiable, as established in GPV. This is similar to the well-known results in the literature on isotonic estimation: without imposing additional smoothness assumptions on the model primitives and without introducing smoothing, one can at most get cube-root-n rate.

In practice it is often useful to conduct joint inference on a set of quantile levels. For example, the test for common values in Haile, Hong, and Shum (2003) and test for different models of entry in Marmer, Shneyerov, and Xu (2013) are characterized by stochastic dominance relations between distributions. The following Corollary shows that the quantile estimator is independent across a fixed vector of quantile levels asymptotically.

Corollary 1. *Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_J < 1$. Then*

$$\mathbb{P}\left(\cap_{j=1,2,\cdots,J}\{n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_j) - Q_v(\alpha_j)) \le z_j\}\right) = \Pi_{j=1,2,\cdots,J}\mathbb{P}\left(C(\alpha_j)\, \mathrm{argmax}_t\{\mathbb{B}_j(t) - t^2\} \le z_j\right),$$

where $\mathbb{B}_j$, $j = 1, 2, \cdots, J$, are independent two-sided Brownian motions.

Proof. See Appendix A.2. □

The result in Corollary 1 does not hold in general for quantile estimations. It is useful when researchers would like to compare multiple quantile levels simultaneously, which in practice is a useful approximation for comparing the whole distribution. One possible choice is the multiple testing procedure of Holm (1979), which controls the familywise error rate of one false rejection. For example, one can calculate the p-vales $\hat{p}_j$, $j = 1, 2, \cdots J$ for each of the J hypothesizes. Rank all the p-values such that $\hat{p}_{(1)} \leq \hat{p}_{(2)} \leq \cdots \leq \hat{p}_{(J)}$. Let α be the significance level. If $\hat{p}_{(1)} \geq \alpha/J$, then no hypothesis is rejected; otherwise the procedure rejects hypothesizes $H_{(1)}, \cdots, H_{(k)}$, where k is the largest integer such that $\hat{p}_{(j)} \leq \alpha/(J - j + 1)$. One important source of conservativeness of Holm's procedure—dependence among p-values—does not rise here.

Theorem 1 also provides a basis for constructing a simple trimming-free smoothed quantile estimator that converges at the optimal rate of GPV under appropriate smoothness conditions as listed in Assumption 3 below. Numerous smooth quantile function estimators have been studied, see, e.g., Nadaraya (1964) for inverting a kernel distribution function estimator, Harrell and Davis (1982) for using generalized order statistics and Cheng (1995) for a Bernstein polynomial estimator. We adopt the kernel estimator used in Yang (1985), which dates back to Parzen (1979). Specifically, for any $0 < \alpha < 1$, let

$$\hat{q}_v(\alpha) = \int_0^1 \frac{1}{h} K\left(\frac{\alpha - u}{h}\right) \widehat{Q}_v(u) du, \tag{4}$$

where h is a bandwidth and $K(\cdot)$ is a kernel with a compact support. Note that by construction, $\hat{q}_v(\cdot)$ is necessarily increasing since $\widehat{Q}_v(\cdot)$ is increasing.

Assumption 3. *The valuation density f is continuously differentiable.*

Assumption 4. *Let K' be the first order derivative of K. Then K satisfies (1) K has compact support and take value zero on the boundary, (2) $\int K'(u)du = \int u^2 K'(u)du = 0$, (3) $\int uK'(u)du = -1$, (4) $\int u^3 K'(u)du \neq 0$.*

Assumption 3 requires same smoothness as in GPV and MS. Assumption 4 is satisfied by commonly used kernel functions such as second order Epanechnikov or Triweight Kernels.

Theorem 2. *Suppose Assumptions 1 to 4 are satisfied, and let $\alpha \in (0,1)$,*

(i) if $nh^5 \to c \in (0,\infty)$, then $\sqrt{nh}(\hat{q}_v(\alpha) - Q_v(\alpha)) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$, where

$$\mathscr{B} = -\left(\frac{\sqrt{c}Q_b''(\alpha)}{3} + \frac{c^2\alpha}{6(I-1)}Q_b'''(\alpha)\right)\int u^3K'(u)du, \quad \mathscr{V} = \frac{\alpha^2}{c(I-1)^2}(Q_b'(\alpha))^2\int K^2(u)du.$$

(ii) if $h = cn^{-r}$ for some $\frac{1}{5} < r < \frac{1}{2}$, then

$$\sqrt{n^{1-r}}(\hat{q}_v(\alpha) - Q_v(\alpha)) \xrightarrow{d} N(0, \mathscr{V}).$$

Proof. See Appendix A.5. □

Note that the variance and bias depends on c analytically. One can estimate the optimal choice of c that minimizes the asymptotic mean squared error, provided that the model has enough smoothness for consistent estimation of $Q_b'''(\cdot)$. Part (ii) of the theorem suggests that we can use under-smoothing to eliminate the asymptotical bias, in the same way as one would do for typically nonparametric estimation of density or regression function. We do not further pursue these issues in this paper.

In the rest of the section, we discuss several interesting extensions of our method.

2.1. **Generalization to procurement auctions.** Our method can be easily adapted to first price procurement auction settings. Suppose that there are I bidders competing for a contract in a first-price sealed bid auction. For each auction, every bidder i simultaneously draws an $i.i.d.$ cost c_i from a common distribution $F(\cdot)$ and submits a bid to maximize his/her expected profit $E[(b_i - c_i)\mathbb{1}(b_i \leq s(\min_{j\neq i} c_j))]$. The lowest bid wins the contract, and the bidder is paid the amount he/she bid.

Differentiating the expected profit with respect to b_i gives the following system of first-order differential equations that define the equilibrium strategy $s(\cdot)$:

$$(b_i - c_i)(I-1)\frac{f[s^{-1}(b_i)]}{[1-F(s^{-1}(b_i))]s'[s^{-1}(b_i)]} = 1,$$

which can be rewritten as

$$c_i = b_i - \frac{1}{I-1}\frac{1-G(b_i)}{g(b_i)}.$$

Therefore, the quantile relationship becomes

$$Q_c(\alpha) = Q_b(\alpha) - (1-\alpha)/[(I-1)g(Q_b(\alpha))],$$

where $Q_c(\cdot)$ represents the cost quantile function. The integrated quantile function becomes

$$C(\beta) \equiv \int_0^\beta Q_c(\alpha)d\alpha = \frac{I-2}{I-1}\int_0^\beta Q_b(\alpha)d\alpha - \frac{1}{I-1}Q_b(\beta)(1-\beta) + \frac{1}{I-1}Q_b(0). \quad (3')$$

To impose the monotonicity constraint, we consider the g.c.m. of the empirical counterpart of the following function:

$$\widetilde{C}(\beta) \equiv C(1-\beta),$$

which is the reflection of the integrated quantile function over the line $\beta = 1/2$. The idea is to utilize the prior information that the maximum possible bid equals the maximum cost in procurement auctions, i.e. $Q_b(1) = Q_c(1)$. As the pseudo values are constructed sequentially, consider the g.c.m. of $\widetilde{C}(\cdot)$ is preferable to $C(\cdot)$. To see this, note that $[\widehat{C}(1) - \widehat{C}(\frac{n-j}{n})]/(j/n) = \frac{I-2}{I-1}\sum_{k=n-j+1}^{N} b_{(k)}/j + \frac{1}{I-1}b_{(N-j)}$ and $[\widehat{C}(1/n) - \widehat{C}(0)]/(1/n) = b_{(1)}$. By definition, the preferred method starts with the largest pseudo valuation $\widehat{c}_{(n)} = \frac{I-2}{I-1}b_{(n)} + \frac{1}{I-1}b_{(n-1)}$. Note that the right-hand side converges to $Q_b(1) = Q_c(1)$ at a fast rate. On the other hand, considering the g.c.m. of $C(\cdot)$, we would start with an estimate of the smallest pseudo valuation $\widehat{c}_{(1)} \leq b_{(1)}$. Although $b_{(1)}$ converges to $Q_b(0)$ at a fast rate, it does not guarantee that $\widehat{c}_{(1)}$ converges to $Q_c(0)$ at a fast rate.

For estimation, we construct a raw estimator $\widetilde{C}_n(\cdot)$ for $\widetilde{C}(\cdot)$ by plugging in the bid quantile estimator. We then take the g.c.m. of $\widetilde{C}_n(\cdot)$. The pseudo cost of the bidder whose bid is the jth highest is constructed as the negative of the right-derivative of the g.c.m. at $\beta = (j-1)/n$, where $j = 1, \ldots, n$. A smooth estimator for the cost quantile function follows naturally: $\hat{q}_c(\alpha) = \int_0^1 \frac{1}{h} K\left(\frac{\alpha - u}{h}\right) \widehat{Q}_c(u) du$. Moreover, we can also apply a kernel density estimator on the sample of pseudo costs: $\hat{f}(c) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{\hat{c}_j - c}{h}\right)$.

2.2. **Estimating the valuation distribution function.** Sometimes an analyst might be more interested in the valuation distribution function than the quantile function. An estimator of valuation distribution function can be obtained by inverting $\widehat{Q}$. In particular, for any $v_0 \in (\underline{v}, \overline{v})$, we can define $\widehat{F}(v_0) = \sup_\alpha\{\widehat{Q}_v(\alpha) \leq v_0\}$. The following corollary establishes the limiting distribution of $\widehat{F}$.[3]

Corollary 2. *Let $v_0 \in (\underline{v}, \overline{v})$ and $\alpha_0 = F(v_0)$. Suppose the conditions of Theorem 1 are satisfied, then for any x, as $n \to \infty$,*

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0) - F(v_0)) < x\right) \to \mathbb{P}\left(f(v_0)C(\alpha_0)\operatorname{argmax}_t\left\{\mathbb{B}(t) - t^2\right\} < x\right),$$

where $C(\alpha_0)$ and $\mathbb{B}$ are as defined in Theorem 1.

Proof. See Appendix A.3. □

To construct an estimator for valuation density, we can first construct a sample of pseudo valuations employing $\widehat{Q}_v(\cdot)$. Let $\hat{v}_j = \widehat{Q}_v(j/n)$, where $j = 1, \ldots, n$. Second, we apply a kernel density estimator on the sample of pseudo values $\{\hat{v}_j\}_{j=1}^{N}$: for $v \in (\underline{v}, \overline{v})$

$$\hat{f}(v) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{\hat{v}_j - v}{h}\right).$$

Since our first step estimator $\widehat{Q}_v(\cdot)$ is tuning-parameter-free, our estimator of the valuation density function requires no trimming and only one tuning parameter h.

[3]Under a similar set of smoothness assumptions to ours, Armstrong (2013a) proposes to estimate the bidding strategy by maximizing the sample analog of the bidder's objective function and subsequently estimates the valuation distribution function at cube-root-n rate. Our approach is based on the integrated-quantile representation of the first order condition and imposes monotonicity restriction. Both estimators are tuning-parameter-free and robust to the degree of smoothness in the model.

2.3. **Incorporating auction level heterogeneity.** In practice, it is of empirical interest to incorporate auction-level observed heterogeneity. In this subsection, we discuss several ways of incorporating such heterogeneity.

In many applications researchers are interested in the valuation quantile/distribution conditional on discrete or discretization of continuous variables. For example, in a procurement auction, researchers may be interested in the valuation (or cost) quantiles of "large" projects, which is defined by whether engineers' estimates (a continuous variable) exceed certain cutoff values. In these cases, our estimation procedure can be directly applied to corresponding subsamples and the estimates can be interpreted as the conditional valuation quantile.

Our method can also be applied together with the homogenization method (see Haile, Hong, and Shum, 2003), one common method of controlling both continuous and discrete observed heterogeneity in the empirical auction literature. The homogenization approach assumes that valuations depend on auction-level characteristics in an additively (or multiplicatively) separable form, which implies that the bids depend on characteristics in the same separable way. Under such an assumption, the effect of auction level characteristics can be controlled by focus on the regression residuals (called homogenized bids) of the original bids (or log of bids in the multiplicative case) on those covariates. Since our estimators converge at rates which are slower than root-n, their asymptotic properties are not affected by the homogenizing step.

If researchers would like to be agnostic about how valuations depend on the continuous covariates X and are interested in the valuations quantiles conditional on a particular realization x, our method can also be applied with modification. In particular, we define the (sample) conditional integrated-quantile functions as

$$V_n(\beta|X=x) = \begin{cases} \frac{I-2}{n(I-1)} \sum_{i=1}^{n\beta} Q_{n,b}(i/n|X=x) + \frac{1}{I-1}\beta Q_{n,b}(n\beta|X=x), & \beta \in \left\{\frac{1}{n}, \cdots, 1\right\} \\ (j-\beta n)V_n\left(\frac{j-1}{n}|X=x\right) + (\beta n - j + 1)V_n\left(\frac{j}{n}|X=x\right), & \beta \in \left(\frac{j-1}{n}, \frac{j}{n}\right) \end{cases}$$

where $Q_{n,b}(\alpha|X=x)$ is a suitable estimator for the conditional quantile function of bids given $X=x$. Note that by definition, $V_n(\cdot|X=x)$ is still a piecewise linear function and its g.c.m., denoted by $\widehat{V}(\cdot|X=x)$, is as easy to compute as in the unconditional case. Our estimator for conditional valuation quantile function, denoted by $\widehat{Q}_v(\cdot|X=x)$, is then defined as the piecewise derivative of $\widehat{V}(\cdot|X=x)$.

For $Q_{n,b}(\alpha|X=x)$, we adopt the local polynomial estimator proposed by GS, who established a uniform Bahadur representation for the conditional quantile function and its derivatives. The limiting results of $\widehat{Q}_v(\alpha|X=x)$ can be derived following the same lines as in Theorem 1.

Corollary 3. *Suppose that (i) X has a continuously differentiable density function which is bounded away from zero over its compact support $\mathscr{X} \subset \mathbb{R}^d$; (ii) for every x, the conditional valuation distribution $F(\cdot|x)$ is continuously differentiable over its compact support $[\underline{v}(x), \overline{v}(x)]$. There exists $\lambda(x) > 0$ such that $\inf_{v\in[\underline{v}(x),\overline{v}(x)]} f(v|x) \geq \lambda(x) > 0$; (iii) the bandwidth for the local polynomial regression is chosen such that $nh^d \to \infty$ and $nh^{d+3} \to 0$. The kernel function is chosen to satisfy Assumption 4. Then for any x in the interior of $\mathscr{X}$ and $\alpha_0 \in (0,1)$,*

$$\sqrt[3]{nh^d}(\widehat{Q}_v(\alpha_0|X=x) - Q_v(\alpha_0|X=x)) = O_p(1).$$

Proof. See Appendix A.4. □

In the presence of covariates, the rate of convergence of our first stage estimator is $\sqrt[3]{nh^d}$, which is slower than the $\sqrt{nh^d}$ rate obtained by GPV and MS under stronger smoothness assumption on $F(\cdot|x)$. This is analogously to the $\sqrt[3]{n}$ versus $\sqrt{n}$ comparison in the homogeneous auction case, with the additional h^d term resulting from conditional on d-dimensional covariates.

3. SIMULATION

To study the finite sample performance of our estimation method, we conduct Monte Carlo experiments. We adopt the setup of the Monte Carlo simulations from MS. The true valuation distribution is

$$F(v) = \begin{cases} 0 & \text{if } v < 0, \\ v^{\gamma} & \text{if } 0 \leq v \leq 1, \\ 1 & \text{if } v > 1, \end{cases}$$

where $\gamma > 0$. Such a choice of private value distributions is convenient since the distributions correspond to linear bidding strategies as:

$$s(v) = \left(1 - \frac{1}{\gamma(I-1)+1}\right) \cdot v. \tag{5}$$

We consider $I = 7$ bidders, $n = 4200$ and $\gamma \in \{0.5, 1, 2\}$. The number of Monte Carlo replications is 1000. For each replication, we first generate randomly n private values from $F(\cdot)$. Second, we obtain the corresponding bids b_i employing the linear bidding strategy (5). Third, we construct a raw estimator $V_n(\cdot)$ for $V(\cdot)$. Let $\widehat{V}(\cdot)$ be the g.c.m. of $V_n(\cdot)$. Fourth, we obtain a sample of pseudo values $\widehat{v}_j$ as the left-derivative of $\widehat{V}(\cdot)$ at j/N and estimate the valuation density function using a kernel estimator.

We compare our method with MS and GPV. For the MS and GPV methods, we use the same setups as in MS: the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidths in estimation of densities. For our method, we also use the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidth in estimation of f: $h = 1.06\widehat{\sigma}_v n^{-1/7}$, where $\widehat{\sigma}_v$ is the estimated standard deviation of the constructed pseudo valuations $\{\widehat{v}_j\}_{j=1}^{N}$.

Table 1 shows the simulation results for density estimation. When the distribution is skewed to the left ($\gamma = 0.5$), our method improves MSE and MAD but seems to produce larger biases near the boundaries. While the MS and GPV methods behave similarly in terms of MSE and MAD, the former seems to produce larger biases. When the distribution is uniform or skewed to the right ($\gamma = 1$ or 2), our method performs similarly to the GPV method, both of which seem perform slightly better than the MS method.

TABLE 1. Simulation Results for Density Estimation

	v		0.2	0.3	0.4	0.5	0.6	0.7	0.8
γ = 0.5	MSE	MS	0.0068	0.0073	0.0103	0.0131	0.0132	0.0171	0.0202
		GPV	0.0056	0.0072	0.0101	0.0132	0.0139	0.0188	0.0218
		Ours	0.0044	0.0057	0.0080	0.0100	0.0109	0.0140	0.0163
	Bias	MS	-0.0041	-0.0019	-0.0086	-0.0029	-0.0159	-0.0156	-0.0185
		GPV	0.0038	0.0018	-0.0034	0.0037	-0.0019	0.0025	0.0072
		Ours	0.0120	0.0043	-0.0016	0.0037	-0.0022	0.0038	0.0056
	MAD	MS	0.0672	0.0689	0.0806	0.0907	0.0908	0.1027	0.1094
		GPV	0.0611	0.0688	0.0800	0.0925	0.0940	0.1106	0.1186
		Ours	0.0543	0.0608	0.0711	0.0806	0.0825	0.0952	0.1030
γ = 1	MSE	MS	0.0036	0.0050	0.0066	0.0076	0.0102	0.0122	0.0148
		GPV	0.0025	0.0035	0.0050	0.0060	0.0082	0.0102	0.0127
		Ours	0.0023	0.0033	0.0049	0.0061	0.0083	0.0102	0.0129
	Bias	MS	0.0003	0.0000	-0.0047	-0.0035	0.0014	-0.0060	-0.0113
		GPV	0.0000	0.0015	-0.0023	-0.0011	0.0053	0.0007	-0.0021
		Ours	0.0000	0.0016	-0.0027	-0.0020	0.0056	0.0007	-0.0026
	MAD	MS	0.0479	0.0557	0.0647	0.0688	0.0800	0.0892	0.0961
		GPV	0.0402	0.0470	0.0563	0.0610	0.0724	0.0806	0.0904
		Ours	0.0389	0.0459	0.0557	0.0615	0.0730	0.0812	0.0901
γ= 2	MSE	MS	0.0016	0.0025	0.0037	0.0063	0.0085	0.0108	0.0154
		GPV	0.0011	0.0016	0.0025	0.0044	0.0060	0.0078	0.0112
		Ours	0.0011	0.0017	0.0028	0.0049	0.0069	0.0091	0.0130
	Bias	MS	-0.0006	-0.0031	-0.0008	-0.0013	-0.0033	-0.0085	-0.0001
		GPV	0.0005	-0.0020	0.0007	0.0002	-0.0004	-0.0044	0.0021
		Ours	0.0006	-0.0019	0.0013	0.0002	-0.0006	-0.0048	0.0020
	MAD	MS	0.0320	0.0394	0.0481	0.0637	0.0739	0.0830	0.1008
		GPV	0.0263	0.0321	0.0396	0.0528	0.0624	0.0707	0.0864
		Ours	0.0266	0.0329	0.0415	0.0555	0.0668	0.0767	0.0929

4. EMPIRICAL ILLUSTRATION

In this section, we implement our method using the California highway procurement data. In particular, we analyze the data used in Krasnokutskaya and Seim (2011). It covers highway and street maintenance projects auctioned by the California Department of Transportation (Caltrans) between January 2002 and December 2005. We focus on the procurement auctions with 2 to 7 bidders. For each auction, the data contain the engineer's estimate of the project's total cost, the type of work involved, the number of days allocated to complete the project, the identity of the bidders and their bids.

Following Haile, Hong, and Shum (2003), we homogenize the bids before implementing our method to control for observable heterogeneity for each sample (with the same number of bidders). In particular, we regress the logarithm of the bid ($logb$) on the logarithm of the engineer's estimate ($logX$), the logarithm of the number of days ($logDays$) and the project type dummies. Table 2 displays the results. The homogenized bids (bid_new) are calculated as the exponential of the differences between the logarithm of the original bids and the demeaned fitted values of the regression. Table 3 displays the mean and standard deviation of the original and homogenized bids.

TABLE 2. Regression Results

	2	3	4	5	6	7
logX	0.978***	0.966***	1.015***	0.957***	0.932***	0.938***
	(34.11)	(56.68)	(50.59)	(51.81)	(49.91)	(56.58)
logDays	0.00650	0.00473	-0.00271	0.0901***	0.138***	0.00430
	(0.15)	(0.25)	(-0.13)	(4.76)	(6.31)	(0.18)
type	Yes	Yes	Yes	Yes	Yes	Yes
n	206	474	564	470	402	252
adj. R^2	0.871	0.906	0.857	0.929	0.930	0.947

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

We estimate a first price auction model with each sample. Figure 1 displays the estimated inverse bidding strategies, the estimated valuation quantile functions without and with smoothing, respectively. The curves represented are: from the sample with 2 bidders (yellow solid line); 3 bidders (magenta dash-dot line); 4 bidders (cyan solid line); 5 bidders (red

TABLE 3. Summary Statistics

	2	3	4	5	6	7	Total
bid	993.8	967.6	757.7	1136.9	990.9	1769.7	1042.8
	(1644.5)	(1935.9)	(843.7)	(4584.7)	(3350.3)	(7288.0)	(3595.9)
bid_new	652.5	587.7	566.3	508.9	464.4	478.5	540.0
	(208.4)	(190.6)	(178.6)	(129.0)	(135.0)	(137.4)	(174.0)
cost	402.1	468.0	477.8	453.8	423.6	441.7	451.5
	(259.6)	(223.6)	(218.9)	(164.4)	(156.3)	(159.5)	(200.0)
profit	250.4	119.7	88.46	55.09	40.79	36.81	88.59
	(75.81)	(77.65)	(60.62)	(49.56)	(42.83)	(46.68)	(83.51)
profit rate	0.439	0.244	0.197	0.136	0.109	0.0978	0.190
	(0.213)	(0.208)	(0.194)	(0.167)	(0.153)	(0.158)	(0.206)

Std. Dev. in parentheses. $profit = bid\_new - cost$. *Profit rate=profit / bid.*

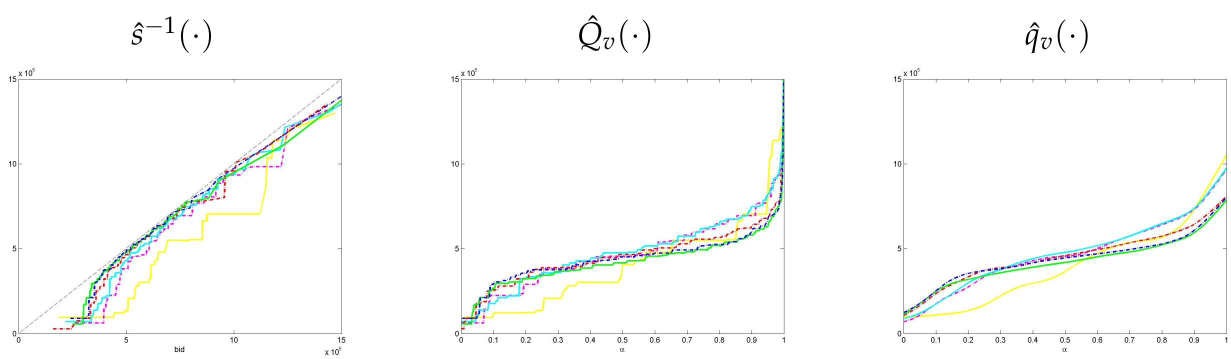

FIGURE 1. Estimation results

dash-dot line); 6 bidders (green solid line); 7 bidders (blue dash-dot line), and the 45-degree line (black dash line).

All inverse bidding strategies are increasing. The valuation quantile functions seem to be close except for $I = 2$. Table 3 displays some summary statistics of the estimated pseudo costs. The auctions with two bidders tend to be less costly to finish in percentage terms. In fact, the generated profit rate is almost twice that of the sample with three bidders. As the auction becomes more competitive when the number of bidders increase from two to seven, the profit rate decreases from 44% to about 10%.

5. Conclusion

This paper considers nonparametric estimation of first-price auction models based on an integrated-quantile representation of the first-order condition. The monotonicity of bidding strategy is imposed in a natural way. We propose two estimators for the valuation quantile function and derive their asymptotics: a non-smoothed estimator that is tuning-parameter-free and a smoothed one that is trimming-free. We show the former is cube-root consistent under weaker smoothness assumptions and the latter achieves the optimal rate of GPV under standard ones. Monte Carlo simulations show our method works well in finite samples. We apply our method to data from the California highway procurements auctions.

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Appendix A. Proof of main results

A.1. **Proof of Theorem 1.** For a generic $c > 0$, let $Z_n(c) = \text{argmin}_{t\in[0,1]}\{V_n(t) - ct\}$. If the argmin is a set, then we take the inf of the set. For any $\alpha_0 \in (0,1)$, by van Es, Jongbloed, and Zuijlen (1998, Theorem 2), the two following events are equivalent

$$Z_n(c) \geq \alpha_0 \Leftrightarrow \widehat{Q}_v(\alpha_0) \leq c.$$

Therefore, we have for a fixed $\alpha_0 \in [0,1)$

$$\begin{aligned}
n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \leq z \Leftrightarrow \widehat{Q}_v(\alpha_0) \leq zn^{-\frac{1}{3}} + Q_v(\alpha_0) \Leftrightarrow Z_n(zn^{-\frac{1}{3}} + Q_v(\alpha_0)) \geq \alpha_0 \\
\overset{i}{\Leftrightarrow} \underset{s\in[0,1]}{\text{argmin}}\{V_n(s) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))s\} \geq \alpha_0 \\
\overset{ii}{\Leftrightarrow} \underset{\{t:\alpha_0+tn^{-\frac{1}{3}}\in[0,1]\}}{\text{argmin}}\{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))(\alpha_0 + tn^{-\frac{1}{3}})\} \geq 0 \\
\overset{iii}{\Leftrightarrow} \underset{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}{\text{argmin}}\{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}} - ztn^{-\frac{2}{3}}\} \geq 0 \\
\overset{iv}{\Leftrightarrow} \underset{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}{\text{argmin}}\{n^{\frac{2}{3}}V_n(\alpha_0 + tn^{-\frac{1}{3}}) - n^{\frac{2}{3}}V_n(\alpha_0) - Q_v(\alpha_0)tn^{\frac{1}{3}} - zt\} \geq 0,
\end{aligned}$$

where (i) holds by definition of Z_n, (ii) holds by changing variable $s = \alpha_0 + tn^{-\frac{1}{3}}$, and (iii) and (iv) hold because the argmin stays unchanged when constants are multiplied or added to, or subtracted from the objective function.

Let $W_n(t) = n^{\frac{2}{3}}\left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}}\right]$, then the above displayed equation reduces to

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \leq z \Leftrightarrow \underset{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}{\text{argmin}}\{W_n(t) - zt\} \geq 0$$

It remains to analyze the asymptotic behavior of $W_n(t)$. Decompose W_n as following

$$\begin{aligned}
W_n(t) = n^{\frac{2}{3}}\left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0)\right] - n^{\frac{2}{3}}\left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0)\right] \\
+ n^{\frac{2}{3}}\left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}}\right].
\end{aligned}$$

The second component equals to $\frac{1}{2}Q_v'(\alpha_0)t^2 + o(1)$ by Assumption 2. By Lemma 3, the first right hand side term converges weakly to $\frac{\alpha_0}{(I-1)\sqrt{g(Q_b(\alpha_0))}}\mathbb{B}(t)$, where $\mathbb{B}$ is a two sided Brownian Motion Therefore, we have

$$W_n(t) \xrightarrow{w} \frac{\alpha_0}{(I-1)g(Q_b(\alpha_0))}\mathbb{B}(t) + \frac{1}{2}Q_v'(\alpha_0)t^2.$$

To simplify the notation, let the constants in front of $\mathbb{B}$ and t^2 be a and b, respectively. Note that $a > 0$ and $b > 0$. By Van Der Vaart and Wellner (1996, Theorem 3.2.2) and the property of Brownian motion,

$$\begin{aligned}
\underset{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}{\operatorname{argmin}} \{W_n(t) - zt\} &\xrightarrow{d} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + bt^2 - zt\} \\
&\sim \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + b(t - \frac{z}{2b})^2 - \frac{z^2}{4b}\} \sim \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + b(t - \frac{z}{2b})^2\} \\
&\sim \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\frac{a}{b}\mathbb{B}(t) + (t - \frac{z}{2b})^2\} \sim \left(\frac{a}{b}\right)^{2/3} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} + \frac{z}{2b}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \le z\right) \to \mathbb{P}\left(\left(\frac{a}{b}\right)^{2/3} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} + \frac{z}{2b} \ge 0\right) \\
&= \mathbb{P}\left(\underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} \ge -\frac{z}{2b}\left(\frac{b}{a}\right)^{2/3}\right) = \mathbb{P}\left(\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t) - t^2\} \le \frac{z}{2b}\left(\frac{b}{a}\right)^{2/3}\right)
\end{aligned}$$

Thus we can conclude that

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \xrightarrow{d} C(\alpha_0)\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t) - t^2\},$$

where $C(\alpha_0) = 2a^{2/3}b^{1/3}$ is a constant depends on α_0.

A.2. **Proof of Corollary 1.** Consider a $J \times 1$ vector of mutually different quantile levels $(\alpha_1, \alpha_2, \cdots, \alpha_J)$, following the arguments in the proof of Theorem 1, the following events are equivalent:

$$\cap_{j=1,2,\cdots,J}\{n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_j) - Q_v(\alpha_j)) \le z_j\} \Leftrightarrow \cap_{j=1,2,\cdots,J}\left\{\underset{t\in[-\alpha_j n^{\frac{1}{3}},(1-\alpha_j)n^{\frac{1}{3}}]}{\operatorname{argmin}}\{W_{jn}(t) - z_j t\} \ge 0\right\},$$

where for $j = 1, 2, \cdots, J$

$$W_{jn}(t) = n^{\frac{2}{3}}\left[V_n(\alpha_j + tn^{-\frac{1}{3}}) - V_n(\alpha_j)\right] - n^{\frac{2}{3}}\left[V(\alpha_j + tn^{-\frac{1}{3}}) - V(\alpha_j)\right] + n^{\frac{2}{3}}\left[V(\alpha_j + tn^{-\frac{1}{3}}) - V(\alpha_j) - Q_v(\alpha_j)tn^{-\frac{1}{3}}\right].$$

Following the same arguments, we have the convergence of each single component:

$$W_{jn}(t) \xrightarrow{w} \frac{\alpha_j}{(I-1)g(Q_b(\alpha_j))}\mathbb{B}_j(t) + \frac{1}{2}Q_v'(\alpha_j)t^2.$$

where $\mathbb{B}_j$ is a two sided Brownian motion. Since $\mathbb{B}_j$ is Gaussian, it remains to find their covariance. Following the arguments in Lemmas 1 to 3 and ignoring the small order terms, we know that for each given t, the joint limiting distribution of W_{jn}, $j = 1, \cdots, J$, is determined by the joint limiting distribution of

$$n^{2/3}\left\{Q_{b,n}(\alpha_j + tn^{-1/3}) - Q_{b,n}(\alpha_j) - Q_b(\alpha_j + tn^{-1/3}) + Q_b(\alpha_j)\right\}, \quad j = 1, \cdots, J,$$

or alternatively, the joint limiting distribution of (for $t > 0$, the case of $t < 0$ is similar)

$$\frac{n^{1/6}}{\sqrt{n}}\sum_i\left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_j) < b_i \le Q_b(\alpha_j + tn^{-1/3})]}{g(Q_b(\alpha_j))}\right), \quad j = 1, \cdots, J,$$

To calculate the limit of covariance of above expression at different quantile levels, it is sufficient to focus on same observation index i since bids are i.i.d.. Since all the α_j are mutually different, the we have for $j \neq j'$, there is $Q_b(\alpha_j) \neq Q_b(\alpha_{j'})$ by the strict monotonicity of Q_b. Therefore,

$$\begin{aligned}&\lim_{n\to\infty} n^{1/3}\mathbb{E}\left[\left(tn^{-1/3} - \mathbf{1}[Q_b(\alpha_j) < b_i \le Q_b(\alpha_j + tn^{-1/3})]\right)\left(tn^{-1/3} - \mathbf{1}[Q_b(\alpha_{j'}) < b_i \le Q_b(\alpha_{j'} + tn^{-1/3})]\right)\right]\\ &= \lim_{n\to\infty}\mathbb{E}\left[-t^2n^{-1/3} + n^{1/3}\mathbf{1}[Q_b(\alpha_{j'}) < b_i \le Q_b(\alpha_{j'} + tn^{-1/3})]\mathbf{1}[Q_b(\alpha_{j'}) < b_i \le Q_b(\alpha_{j'} + tn^{-1/3})]\right] = 0.\end{aligned}$$

Therefore, we can conclude that $\mathbb{B}_j$ are asymptotically uncorrelated and hence independent.

Let constants (a_j, b_j) be defined as

$$a_j = \frac{\alpha_j}{(I-1)g(Q_b(\alpha_j))}, \quad b_j = \frac{1}{2}Q_v'(\alpha_j).$$

Then we have the joint limiting distribution be given by

$$
\begin{aligned}
&\mathbb{P}\left(\cap_{j=1,2,\cdots,J}\{n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_j)-Q_v(\alpha_j))\le z_j\}\right)\\
&\qquad\to\mathbb{P}\left(\cap_{j=1,2,\cdots,J}\left\{\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}_j(t)-t^2\}\le\frac{z_j}{2b_j}\left(\frac{b_j}{a_j}\right)^{2/3}\right\}\right)\\
&\qquad\qquad=\Pi_{j=1,\cdots J}\mathbb{P}\left(\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}_j(t)-t^2\}\le\frac{z_j}{2b_j}\left(\frac{b_j}{a_j}\right)^{2/3}\right).
\end{aligned}
$$

A.3. **Proof to Corollary 2.** Recall that $\widehat{F}(v_0)=\sup_\alpha\{\widehat{Q}_v(\alpha)\le v_0\}$. Consistency of $\widehat{F}(v_0)$ holds by the consistency of $\widehat{Q}_v$ and the continuity of the sup operator. It remains to work out the convergence rate and limiting distribution. Let $Z=\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t)-t^2\}$. Now,

$$
\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0)-F(v_0))<x\right)=\mathbb{P}\left(\widehat{F}(v_0)<n^{-1/3}x+F(v_0)\right)
$$

Note that the event $\{\widehat{F}(v_0)<n^{-1/3}x+F(v_0)\}$ is equivalent to $\{v_0<\widehat{Q}_v(n^{-1/3}x+F(v_0))\}$. Using the fact that $F(v_0)=\alpha_0$, $Q_v(\alpha_0)=v_0$, and $(Q_v'(\alpha_0))^{-1}=f(v_0)$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\widehat{F}(v_0)<n^{-1/3}x+F(v_0)\right)=\mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x+F(v_0))>v_0)\right)\\
&\quad=\mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x+\alpha_0)-Q_v(n^{-1/3}x+\alpha_0)>v_0-Q_v(n^{-1/3}x+\alpha_0)\right)\\
&\quad=\mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x+\alpha_0)-Q_v(n^{-1/3}x+\alpha_0)>-n^{-1/3}Q_v'(\alpha_0)x+O(n^{-2/3}))\right)\\
&\quad=\mathbb{P}\left(f(v_0)n^{1/3}(\widehat{Q}_v(n^{-1/3}x+\alpha_0)-Q_v(n^{-1/3}x+\alpha_0))<x+O(n^{-1/3}))\right)
\end{aligned}
$$

Repeat the proof of Theorem 1 shows that for each x, $n^{1/3}(\widehat{Q}_v(n^{-1/3}x+\alpha_0)-Q_v(n^{-1/3}x+\alpha_0))$ has the same limiting distribution as $n^{1/3}(\widehat{Q}_v(\alpha_0)-Q_v(\alpha_0))$. Therefore, we have

$$
\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0)-F(v_0))<x\right)\to\mathbb{P}\left(f(v_0)C(\alpha_0)Z<x\right).
$$

A.4. **Proof to Corollary 3.** We give the sketch of the proof for brevity. Let γ_n be a deterministic diverging sequence whose rate will be determined later. For a given x, define

$$
W_n(t|x)=\gamma_n^2\left[V_n(\alpha_0+t\gamma_n^{-1}|x)-V_n(\alpha_0|x)-Q_v(\alpha_0|x)t\gamma_n^{-1}\right].
$$

Following the same argument as in Theorem 1, we have

$$\gamma_n^{-1}(\widehat{Q}_v(\alpha_0|x) - Q_v(\alpha_0|x)) \leq z \Leftrightarrow \underset{t\in[-\alpha_0\gamma_n,(1-\alpha_0)\gamma_n]}{\operatorname{argmin}} \{W_n(t|x) - zt\} \geq 0$$

Then we conduct the same decomposition:

$$W_n(t|x) = \underbrace{\gamma_n^2 \left[V_n(\alpha_0 + t\gamma_n^{-1}|x) - V_n(\alpha_0|x)\right] - \gamma_n^2 \left[V(\alpha_0 + t\gamma_n^{-1}|x) - V(\alpha_0|x)\right]}_{\equiv W_n^A(t)} + \underbrace{\gamma_n^2 \left[V(\alpha_0 + t\gamma_n^{-1}|x) - V(\alpha_0|x) - Q_v(\alpha_0|x)t\gamma_n^{-1}\right]}_{=\frac{1}{2}Q_v'(\alpha_0|x)t^2+o(1)}.$$

It remains to analyze the asymptotic behavior of W_n^A. It can be observed from the definition of $V_n(\cdot|x)$ that for any $\tau \in (0,1)$,

$$V_n(\tau|x) = \frac{I-2}{I-1}\int_0^\tau Q_{n,b}(t|x)dt + \frac{1}{I-1}\tau Q_{n,b}(\tau|x) + O(1/n),$$

where $Q_{n,b}(\tau|x)$ is chosen to be the local polynomial estimator of Guerre and Sabbah (2012), whose Assumptions X, F and K can be verified to hold in our context. Since we do not have to estimate the quantile derivatives, we choose the order ν of the polynomial as $\nu = 0$. Using Guerre and Sabbah (2012)'s uniform Bahadur representation, we have for any $\tau \in (0,1)$,

$$Q_{n,b}(\tau|x) - Q_b(\tau|x) = \frac{\beta_n(\tau)}{(nh^d)^{1/2}} + O\left(h^2\right) + O_p\left(\frac{\log n}{nh^d}\right)^{3/4},$$

where the first right hand side (RHS) is the first order approximation, the second RHS term is the bias and its order is determined by the twice continuous differentiability of Q_b, and the third RHS term is the Bahadur representation error, and β_n is defined as

$$\beta_n(\tau) = J_n^{-1}\frac{2}{(nh^d)^{1/2}}\sum_i^n \{\mathbf{1}[b_i \leq Q_b^*(\tau|x)] - \tau\} K\left(\frac{X_i - x}{h}\right),$$

where $J_n \xrightarrow{p} J$ for some constant, $K(\cdot)$ is the kernel function and Q_b^* is the argmin of the population criterion function.

Now take $\gamma_n = (nh^d)^{1/3}$, then $O(\gamma_n^2 h^2) = o(1)$ since h is chosen such that $nh^{d+3} \to 0$, and $\left(\frac{\gamma_n^2 \log n}{nh^d}\right)^{3/4} = o_p(1)$ since $nh^d \to \infty$. Note that the bias is eliminated by under-smoothing.

Following similar argument as in Lemmas 1 to 3, the limiting behavior of $W_n^A(t)$ when $t \geq 0$ (the case of $t < 0$ similar) depends on the following dominant term (up to additive asymptotically negligible and some multiplicative constant terms):

$$\frac{1}{\sqrt{nh^d}} \sum_i^n \underbrace{(nh^d)^{1/6} \left\{ t(nh^d)^{-1/3} - \mathbf{1}[Q_b^*(\alpha_0|x) < b_i \leq Q_b^*(\alpha_0 + t(nh^d)^{-1/3}|x)] \right\}}_{\equiv \xi_i(t)} K\left(\frac{X_i - x}{h}\right).$$

Guerre and Sabbah (2012, Lemma A.1) shows that $\xi_i(t)$ has zero mean. Furthermore, for arbitrary $t, s > 0$,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[\xi_i(t)\xi_i(s)] = \lim_{n\to\infty} \mathbb{E}\{\mathbb{E}[\xi_i(t)\xi_i(s)|x]\} \\ = \lim_{n\to\infty} \mathbb{E}\left\{ K^2\left(\frac{X_i - x}{h}\right) [\min\{t, s\} + O(\gamma_n^{-1})] \right\} \\ = h^d f_X(x) \min\{t, s\} \int K^2(u)du + o(h^d),\end{aligned}$$

it follows that $\frac{1}{\sqrt{nh^d}} \sum_i \xi_i(t)$ converges in distribution to normal for each t and given $\xi_i(t)$ is sum of indicator functions, $\frac{1}{\sqrt{nh^d}} \sum_i \xi_i(\cdot)$ weakly converge to a constant multiplied by a Brownian motion process. The rest of the proof follows similarly from Theorem 1.

A.5. **Proof of Theorem 2.** For notation simplicity, let $K_h(\cdot) = (1/h)K(\cdot/h)$. Then

$$\begin{aligned}\hat{q}_v(\alpha) = \int K_h(\alpha - u)\, d\widehat{V}(u) = \int K_h(\alpha - u)\, dV_n(u) + \int K_h(\alpha - u)\, d(\widehat{V} - V_n)(u) \\ = \int K_h(\alpha - u)\, dV_n(u) + \frac{1}{h}\int K_h'(\alpha - u)(\widehat{V}(u) - V_n(u))du \\ = \int K_h(\alpha - u)\, dV_n(u) + \frac{1}{h}\int K_h'(t)(\widehat{V}(\alpha + ht) - V_n(\alpha + ht))dt \\ = \int K_h(\alpha - u)\, dV_n(u) + O_p((n/\log n)^{-2/3}/h) \qquad (6)\end{aligned}$$

where the third inequality holds by integration by parts, and the last equality holds by Lemma 8. It is then sufficient to focus on the first right hand side term. Since $Q_{v,n}$ is piecewise flat and is left-continuous, we have

$$\int K_h(\alpha-u)\,dV_n(u)-Q_v(\alpha)=\int K_h(\alpha-u)\,Q_{v,n}(u)du-Q_v(\alpha)$$

$$=\underbrace{\sum_{i=1}^{n}b_{(i)}\int_{\frac{i-1}{n}}^{\frac{i}{n}}K_h(\alpha-u)\,du-Q_b(\alpha)}_{A_n(\alpha)}+\frac{1}{I-1}\underbrace{\left(\sum_{i=1}^{n}(i-1)(b_{(i)}-b_{(i-1)})\int_{\frac{i-1}{n}}^{\frac{i}{n}}K_h(\alpha-u)\,du-\frac{\alpha}{g(Q_b(\alpha))}\right)}_{B_n(\alpha)}.$$

$A_n(\alpha)$ is the standard smooth quantile estimator. Yang (1985, Theorem 1) shows that when $nh^5\to c$, $\sqrt{nh}A_n(\alpha)\xrightarrow{p}\sqrt{c}Q_b''(\alpha)\int u^2K(u)du=-\frac{\sqrt{c}}{3}Q_b''(\alpha)\int u^3K'(u)du$, and when $nh^5\to 0$, $\sqrt{nh}A_n(\alpha)\xrightarrow{p}0$. On the other hand, Lemma 10 shows that when $nh^5\to c$, $\sqrt{nh}(\tilde{B}_n(\alpha)-\frac{\alpha}{g(Q_b(\alpha))})\xrightarrow{d}N(\mathscr{B},\mathscr{V})$, where

$$\mathscr{B}=-\frac{c^2\alpha}{6(I-1)}Q_b'''(\alpha)\int u^3K'(u)du\quad \mathscr{V}=\frac{\alpha^2}{c(I-1)^2}(Q_b'(\alpha))^2\int K^2(u)du,$$

and when $nh^5\to 0$, $\sqrt{nh}(\tilde{B}_n(\alpha)-\frac{\alpha}{g(Q_b(\alpha))})\xrightarrow{d}N(0,\mathscr{V})$. This establishes the conclusion of Theorem 2.

Appendix B. Lemmas for Theorem 1

Lemma 1. *Suppose that Assumptions 1 and 2 hold, then for any $\alpha_0\in(0,1)$ and uniformly over $t\in\mathscr{T}$, where $\mathscr{T}$ is compact,*

$$n^{2/3}\left\{\int_{\alpha_0}^{\alpha_0+t/n^{1/3}}Q_{b,n}(\tau)d\tau-\int_{\alpha_0}^{\alpha_0+t/n^{1/3}}Q_b(\tau)d\tau\right\}\xrightarrow{p}0.$$

Proof. Assumption 2 implies that Q_b is twice continuously differentiable (see Guerre, Perrigne, and Vuong, 2000, Proposition 1-(iv)). By the Bahadur representation for quantile functions (see, e.g. Bahadur, 1966; Kiefer, 1967), we know that uniform in $\tau\in[\delta,1-\delta]$,

$$Q_{b,n}(\tau)-Q_b(\tau)=\frac{\tau-\frac{1}{n}\sum_i\mathbf{1}[b_i\le Q_b(\tau)]}{g(Q_b(\tau))}+O_{a.s.}\left(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}\right).$$

Since $\alpha_0 \in (0,1)$, we have

$$n^{2/3} \int_{\alpha_0}^{\alpha_0+t/n^{1/3}} (Q_{b,n}(\tau) - Q_b(\tau))\, d\tau = n^{2/3} \int_{\alpha_0}^{\alpha_0+t/n^{1/3}} \left(\frac{\tau - \frac{1}{n}\sum_i \mathbf{1}[b_i \le Q_b(\tau)]}{g(Q_b(\tau))} \right) d\tau + o_p(1)$$
$$= n^{2/3} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \left(F(u) - \frac{1}{n}\sum_i \mathbf{1}[b_i \le u] \right) du + o_p(1)$$
$$= \frac{1}{\sqrt{n}} \sum_i n^{1/6} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} (F(u) - \mathbf{1}[b_i \le u])\, du + o_p(1)$$
$$= \frac{1}{\sqrt{n}} \sum_i \xi_n(b_i, t) + o_p(1),$$

where $\xi_n(b_i,t) = n^{1/6} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} (F(u) - \mathbf{1}[b_i \le u])\, du$. It is sufficient to show that $\frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t)$ converges uniformly to zero in probability.

Note that $\mathbb{E}[\xi_n(b_i,t)] = 0$ and the summand are i.i.d.. For each n, define a class of functions indexed by t: $\Xi_n \equiv \{\xi_n(\cdot,t) : t \in \mathscr{T}\}$. Then we can have the following observations.

(i) Let $t^* = \operatorname{argmax}_{t\in\mathscr{T}} |Q_b(\alpha_0 + t/n^{1/3}) - Q_b(\alpha_0)|$ and let $\bar{\bar{\xi}}(b) \equiv n^{1/6}|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)|(F(u) - \mathbf{1}[b \le u])$. Then $\bar{\bar{\xi}}(b)$ is an envelope function for Ξ_n. We also have $\mathbb{E}\bar{\bar{\xi}}^2(b) = O(1)$ since $|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)| = O(n^{-1/3})$.

(ii) For any $\epsilon > 0$, we have $\mathbb{E}[\bar{\bar{\xi}}^2(b)\mathbf{1}[\bar{\bar{\xi}}(b) > \epsilon\sqrt{n}]] = o(1)$. This is because $\bar{\bar{\xi}}(b) > \epsilon\sqrt{n}$ if and only if $|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)|(F(u) - \mathbf{1}[b \le u]) > \epsilon n^{1/3}$ and the latter is a probability event with probability approaches zero, whereas $\mathbb{E}[\bar{\bar{\xi}}^2]$ is bounded.

(iii) For any $\epsilon_n \downarrow 0$, there is $\sup_{(t,s)\in\mathscr{T}^2:|t-s|\le\epsilon_n} \mathbb{E}\{\xi_n(b,t) - \xi_n(b,s)\}^2 = o(1)$. To verify this claim, assume without loss of generality that $t > 0$ and $s < 0$. Then $\xi_n(b,t) - \xi_n(b,s) = n^{1/6} \int_{Q_b(\alpha_0+s/n^{1/3})}^{Q_b(\alpha_0+t/n^{1/3})} (F(u) - \mathbf{1}[b_i \le u])\, du$ and almost surely

$$\{\xi_n(b,t) - \xi_n(b,s)\}^2 = n^{1/3} \left\{ \int_{Q_b(\alpha_0+s/n^{1/3})}^{Q_b(\alpha_0+t/n^{1/3})} (F(u) - \mathbf{1}[b_i \le u])\, du \right\}^2$$
$$\le n^{1/3} \left\{ \int_{Q_b(\alpha_0+s/n^{1/3})}^{Q_b(\alpha_0+t/n^{1/3})} |F(u) - \mathbf{1}[b_i \le u]|\, du \right\}^2$$
$$\le 4n^{1/3} \left\{ Q_b(\alpha_0 + t/n^{1/3}) - Q_b(\alpha_0 + s/n^{1/3}) \right\}^2 = n^{-1/3}O(|t-s|),$$

where the second inequity holds because $|F(u) - \mathbf{1}[b_i \leq u]| \leq \sup_u F(u) + 1 \leq 2$. The claim is therefore verified.

(iv) Let $\mathcal{N}(\epsilon, \Xi_n, \mathbb{L}^2(\mathcal{P}))$ be the $\mathbb{L}^2$-covering number for Ξ_n with respect to probability measure $\mathcal{P}$, then for every $\epsilon_n \downarrow 0$, we have $\sup_{\mathcal{P}^*} \int_0^{\epsilon_n} \sqrt{\log \mathcal{N}(\epsilon \|\bar{\xi}(b)\|_{\mathcal{P}^*,2}, \Xi_n, \mathbb{L}^2(\mathcal{P}^*))} d\epsilon = o(1)$. This claim holds by observing that $\xi_{b,t}$ is continuously differentiable with respect to t and hence Ξ_n belongs to the parametric class (see Van der Vaart, 2000, Example 19.7), which implies the convergences of the integral.

(v) We derive the limit of the covariance function. Take $t, s \in \mathcal{T}$,

$$
\begin{aligned}
\mathbb{E}[\xi_n(b_i,t)\xi_n(b_i,s)] &= \mathbb{E}\left[n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \mathbf{1}[b_i \leq u] du \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})} \mathbf{1}[b_i \leq u] du\right] + o(1) \\
&= n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})} \mathbb{E}\left\{\mathbf{1}[\min\{u,v\} \geq b_i]\right\} dudv + o(1) \\
&= n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})} G(\min\{u,v\}) dudv \to 0,
\end{aligned}
$$

where G is the c.d.f. of the bid distribution. Therefore, $H(t,s) \equiv \lim_{n\to\infty} \mathbb{E}[\xi_n(b_i,t)\xi_n(b_i,s)] = 0$ for any $t, s \in \mathcal{T}$.

Based on (i)-(v) and Van Der Vaart and Wellner (1996, Theorem 2.11.22), $\frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t)$ converges weakly to a zero mean Gaussian process $\mathbb{G}$ with sample path define on $\mathcal{T}$ and with covariance function $H(t,s)$. By the property of Gaussian process, $H(t,s) = 0$ implies that the limit process $\mathbb{G}(t) = 0$ for all t almost surely. Because the mapping $\sup_t \mathbb{G}(\cdot) \to \mathbb{R}$ (from the set of continuous functions defined on compact set to $\mathbb{R}$) is continuous with respect to the sup-norm, we can further apply the continuous mapping theorem and have

$$
\frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,\cdot) \xrightarrow{w} \mathbb{G} \Rightarrow \sup_t \frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t) \xrightarrow{d} \sup_t \mathbb{G}(t) = 0 \Rightarrow \sup_t \frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t) \xrightarrow{p} 0.
$$

The conclusion of the Lemma holds.

Lemma 2. *Suppose that Assumptions 1 and 2 hold, then*

$$
n^{2/3}\alpha_0\left\{Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0)\right\} \xrightarrow{w} \frac{\alpha_0}{g(Q_b(\alpha_0))}\mathbb{B}(t),
$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. By Van Der Vaart and Wellner (1996, Theorem 1.6.1), it is sufficient to show the result holds for a sequence of compact sets $\mathscr{T}_1 \subseteq \mathscr{T}_2 \subseteq \cdots \subseteq \mathscr{T}_k \subseteq \cdots$ such that $0 \in \mathscr{T}_1$ and $\cup_{k=1}^{\infty} \mathscr{T}_k = \mathbb{R}$. Denote $\mathscr{T}_k^+ = \mathscr{T}_k \cap \mathbb{R}^+$ and $\mathscr{T}_k^- = \mathscr{T}_k \cap \mathbb{R}^-$. Given Assumption 2, we can apply Bahadur representation again (see Lemma 1) and know that uniform in τ,

$$Q_{b,n}(\tau) - Q_b(\tau) = \frac{\tau - \frac{1}{n}\sum_i \mathbf{1}[b_i \leq Q_b(\tau)]}{g(Q_b(\tau))} + O_{a.s.}(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}.$$

We consider $t \geq 0$ first. Let $r_{1n} = O_{a.s.}(n^{-1/12}(\log n)^{1/2}(\log\log n)^{1/4}$, we have uniformly in $t \in \mathscr{T}_k^+$,

$$\begin{aligned}
&n^{2/3}\left\{Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0)\right\} \\
&= \frac{n^{1/6}}{\sqrt{n}}\sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 - \mathbf{1}[b_i \leq Q_b(\alpha_0)]}{g(Q_b(\alpha_0))}\right) + r_{1n} \\
&\qquad = \frac{n^{1/6}}{\sqrt{n}}\sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_0) < b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))}\right) + r_{1n} + r_{2n},
\end{aligned}$$

where

$$\begin{aligned}
r_{2n} &= \frac{n^{1/6}}{\sqrt{n}}\sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))}\right) \\
&= n^{1/6}\left(\frac{1}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{1}{g(Q_b(\alpha_0))}\right)\frac{1}{\sqrt{n}}\sum_i \xi_i = n^{1/6}O(n^{-1/3})O_p(1) = o_p(1),
\end{aligned}$$

where $\xi_i = \alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]$. For the leading term, it is can be shown by standard method (e.g. Kim and Pollard, 1990) that

$$\frac{n^{1/6}}{\sqrt{n}}\sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_0) < b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))}\right) \xrightarrow{w} \frac{1}{g(Q_b(\alpha_0))}\mathbb{B}(t),$$

where $\mathbb{B}$ is a Brownian motion over a sequence of compact sets $\mathscr{T}_1^+ \subseteq \mathscr{T}_2^+ \subseteq \cdots \subseteq \mathscr{T}_k^+ \subseteq \cdots$.

When $t < 0$, we have uniformly in $t \in \mathscr{T}_k^-$,

$$n^{2/3}\left\{Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0)\right\}$$
$$= \frac{n^{1/6}}{\sqrt{n}}\sum_i\left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 - \mathbf{1}[b_i \le Q_b(\alpha_0)]}{g(Q_b(\alpha_0))}\right) + \tilde{r}_{1n}$$
$$= \frac{n^{1/6}}{\sqrt{n}}\sum_i\left(\frac{tn^{-1/3} + \mathbf{1}[Q_b(\alpha_0 + tn^{-1/3}) < b_i \le Q_b(\alpha_0)]}{g(Q_b(\alpha_0))}\right) + \tilde{r}_{1n} + \tilde{r}_{2n},$$

where the two asymptotically negligible terms $\tilde{r}_{1n}$ and $\tilde{r}_{2n}$ are analogously defined as r_{1n} and r_{2n} in the proof of the case $t \ge 0$, respectively. The convergence result holds analogously over a sequence of compact sets $\mathscr{T}_1^- \subseteq \mathscr{T}_2^- \subseteq \cdots \subseteq \mathscr{T}_k^- \subseteq \cdots$.

The conclusion follows by combining the results for both $t \ge 0$ and $t < 0$.

Lemma 3. *Suppose that Assumptions 1 and 2 hold, then*

$$n^{\frac{2}{3}}\left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0)\right] - n^{\frac{2}{3}}\left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0)\right] \xrightarrow{w} \frac{\alpha_0}{(I-1)g(Q_b(\alpha_0))}\mathbb{B}(t)$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. Recall that for any $\tau \in (0,1)$,

$$V_n(\tau) = \frac{1}{n}\frac{I-2}{I-1}\sum_i b_i\mathbf{1}[b_i \le Q_{b,n}(\tau)] + \frac{1}{I-1}\tau Q_{b,n}(\tau) + O_p(1/n)$$
$$\equiv \frac{I-2}{I-1}V_{1n}(\tau) + \frac{1}{I-1}V_{2n}(\tau) + O_p(1/n).$$

Likewise,

$$V(\tau) = \frac{I-2}{I-1}\int_0^\tau Q_v(t)dt + \frac{1}{I-1}\tau Q_b(\tau) \equiv \frac{I-2}{I-1}V_1(\tau) + \frac{1}{I-1}V_2(\tau).$$

The part associates with V_{1n}, that is, $n^{\frac{2}{3}}\left[V_{1n}(\alpha_0+tn^{-\frac{1}{3}})-V_{1n}(\alpha_0)\right]-n^{\frac{2}{3}}\left[V_1(\alpha_0+tn^{-\frac{1}{3}})-V_1(\alpha_0)\right]$ converges in probability to zero by Lemma 1. For the part associated with V_{2n}, note that

$$
\begin{aligned}
n^{\frac{2}{3}}(I-1)&\left[V_{2n}(\alpha_0+tn^{-\frac{1}{3}})-V_{2n}(\alpha_0)\right]-n^{\frac{2}{3}}\left[V_2(\alpha_0+tn^{-\frac{1}{3}})-V_2(\alpha_0)\right]\\
&=n^{2/3}Q_{b,n}(\alpha_0+tn^{-1/3})(\alpha_0+tn^{-1/3})-n^{2/3}Q_{b,n}(\alpha_0)\alpha_0-n^{2/3}Q_b(\alpha_0\\
&\qquad+tn^{-1/3})(\alpha_0+tn^{-1/3})+n^{2/3}Q_b(\alpha_0)\alpha_0\\
&=n^{2/3}\alpha_0\left\{Q_{b,n}(\alpha_0+tn^{-1/3})-Q_{b,n}(\alpha_0)-Q_b(\alpha_0+tn^{-1/3})+Q_b(\alpha_0)\right\}\\
&\qquad+n^{1/3}t\left\{Q_{b,n}(\alpha_0+tn^{-1/3})-Q_b(\alpha_0+tn^{-1/3})\right\}
\end{aligned}
$$

The second right hand side term, for $|t|<K$, is uniformly bounded by order $n^{1/3}\times n^{-1/2}\times O_p(1)\xrightarrow{p}0$. The first right hand side term is dealt with by Lemma 2.

APPENDIX C. LEMMAS FOR THEOREM 2

We introduce some notation. Let k_n be a sequence of integers such that $k_n\to\infty$ and $n/k_n\to\infty$. Without loss of generality we assume k_n divides n and let $\ell_n=n/k_n$. We therefore can divide $[0,n]$ into k_n equal size intervals with each interval contains ℓ_n consecutive integers. Let $\{s_i,i=1,2,\cdots,k_n\}$ be the set of upper boundary of those intervals such that $s_i=i\ell_n$.

For $(i-1)\ell_n\le s<i\ell_n$, $i=1,2,\cdots,k_n$, define

$$L(s)=\frac{s-(i-1)\ell_n}{\ell_n}V\left(\frac{i}{n}\right)+\frac{i\ell_n-s}{\ell_n}V\left(\frac{i-1}{n}\right),$$

and

$$L_n(s)=\frac{s-(i-1)\ell_n}{\ell_n}V_n\left(\frac{i}{n}\right)+\frac{i\ell_n-s}{\ell_n}V_n\left(\frac{i-1}{n}\right),$$

That is, L and L_n are the linear interpolation of V and V_n on k_n knots $\{s_1/n,s_2/n,\cdots,s_{k_n}/n\}$, respectively. Note that since V is convex under H_0, L is necessarily convex. However L_n may not be convex since V_n is not necessarily convex. Let A_n be the event such that L_n is convex. Since L_n is convex if and only if each segment is convex, the complement of A_n can be written as

$$A_n^c = \bigcup_{i=2}^{k_n-1} \left\{ V_n\left(\frac{(i-1)\ell_n}{n}\right) + V_n\left(\frac{(i+1)\ell_n}{n}\right) < 2V_n\left(\frac{i\ell_n}{N}\right) \right\}$$
$$= \bigcup_{i=2}^{k_n} \left\{ V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right) \right.$$
$$\left. + \Delta_n\left(\frac{(i-1)\ell_n}{n}\right) + \Delta_n\left(\frac{(i+1)\ell_n}{n}\right) - 2\Delta_n\left(\frac{i\ell_n}{n}\right) < 0 \right\},$$

where $\Delta_n \equiv V_n - V$.

Lemmas 4 to 8 shows that the sup distance between V_n and $\widehat{V}$ is small, which we adapted from Pal and Woodroofe (2006). Lemmas 9 and 10 establish the limiting distribution of the smoothed estimator.

Lemma 4. *Suppose that Assumption 3 is satisfied, then there exists a positive c_1 such that* $\min_{i=2,\cdots,k_n-1} |V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right)| \geq \frac{c_1}{k_n^2}$.

Proof. By Assumption 3, there exists $c_1 > 0$ such that $Q_v'(\alpha) \geq c_1 > 0$ for all $\alpha \in [0,1]$. Then we have

$$V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right)$$
$$= \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} Q_v(\alpha)d\alpha - \int_{\frac{(i-1)\ell_n}{n}}^{\frac{i\ell_n}{n}} Q_v(\alpha)d\alpha \geq \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} \left[Q_v(\alpha) - Q_v\left(\frac{i\ell_n}{n}\right)\right] d\alpha$$
$$= \frac{\ell_n}{n}\left[Q_v(\alpha_n^*) - Q_v\left(\frac{i\ell_n}{n}\right)\right] \geq c_1\frac{\ell_n^2}{n^2} = \frac{c_1}{k_n^2}. \quad \square$$

Lemma 5. *Let $\|\cdot\|$ denote the sup norm. Conditional on A_n, there is*

$$\|V_n - \widehat{V}\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|.$$

Proof. By Kiefer and van Wolfowitz (1976), for any convex function m, $\|\widehat{V} - m\| \leq \|V_n - m\|$. Therefore,

$$\|V_n - \widehat{V}\| \leq \|V_n - L_n\| + \|L_n - \widehat{V}\| \leq 2\|V_n - L_n\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|. \quad \square$$

Lemma 6. *Suppose that Assumption 3 is satisfied, then there exists $c_3 > 0$ such that for all $s \in [0, n]$,*

$$0 \le L(s) - V(s) \le \frac{c_3}{k_n^2}.$$

Proof. $L(s) > V(s)$ follows immediately by the convexity of V. The other inequality holds follows from a similar argument as in Lemma 4 and the fact that $Q_v'(\alpha)$ is bounded from above uniformly.

Lemma 7. *Suppose that Assumptions 1 and 3 is satisfied, then*

$$\|V_n - L_n - V + L\| = O_p\left(\sqrt{\frac{\log k_n}{nk_n}}\right) + O_p\left(\frac{\log n}{n}\right).$$

Proof. Define function V_P such that $V_P(j/n) = V(j/n)$ for each j/n and otherwise equals to its own interpolation. It is obvious that $\|V_P - V\| = O(1/n)$. It is then sufficient to focus on $V_n - L_n - V_P + L$. Note that all four functions are piece-wise linear, and so does there linear combinations. Therefore, the sup must be achieved at some knot(s). Based on this observations, we can write

$$\begin{aligned}
&\|V_n - L_n - V_P + L\| \\
&\qquad = \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_n(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_n((i-1)/n) \right|,
\end{aligned}$$

where for $t \in [0, 1]$,

$$\begin{aligned}
&\Delta_n(t) = V_n(t) - V_P(t) = V_n(t) - V(t) + O(1/n) \\
&\qquad\qquad = \frac{I-2}{I-1}\underbrace{\left\{\sum_{i=1}^{[tn]} \frac{b_{(i)}}{n} - \int_0^t Q_b(\alpha)d\alpha\right\}}_{\Delta_A(t)} + \frac{1}{I-1}\underbrace{\left\{\frac{[tn]}{n}b_{(j)} - tQ_b(t)\right\}}_{\Delta_B(t)} + O(1/n)
\end{aligned}$$

where $[x]$ denotes the integer part of x. Note that Δ_A is an integrated quantile process. By Tse (2009, Theorem 2.1), there exists a Gaussian process $\mathbb{G}_n$ and Brownian bridge $\mathbb{B}_n^A$ defined on proper measurable space such that for any $\tau < 1/6$,

$$\|\sqrt{n}\Delta_A - \psi_n\| \overset{a.s.}{=} O(n^{-\tau}),$$

where $\psi_n(t) = \mathbb{G}_n(t) + \int_0^t \mathbb{B}_n^A(u)dQ_b(u)$. On the other hand, by Csorgo and Revesz (1978, Theorem 6), there exists a sequence of Brownian bridge B_n such that $\sup_{\delta_n \le t \le 1-\delta_n} |g(Q_b(t))\sqrt{n}\Delta_B(t) - B_n(t)| \overset{a.s.}{=} O_p(n^{-1/2}\log n)$. We can then conclude

$$
\begin{aligned}
&\|V_n - L_n - V_P + L\| \\
&\quad \le \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_A(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_A(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_A((i-1)/n) \right| \\
&+ \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_B(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_B(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_B((i-1)/n) \right| + O_p(1/n) \\
&\overset{d}{=} \frac{1}{\sqrt{n}} \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \psi_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\psi_n(i/n) - \frac{i\ell_n - j}{\ell_n}\psi_n((i-1)/n) \right| + O_p(n^{-\tau-1/2}) \\
&+ \frac{1}{\sqrt{n}} \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| B_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}B_n(i/n) - \frac{i\ell_n - j}{\ell_n}B_n((i-1)/n) \right| + O_p(\log n/n) \\
&\le \frac{1}{\sqrt{n}} \sup_{0 \le t-s \le \frac{1}{k_n}} |\psi_n(t) - \psi_n(s)| + \frac{1}{\sqrt{n}} \sup_{0 \le t-s \le \frac{1}{k_n}} |B_n(t) - B_n(s)| + O_p(\log n/n) + O_p(n^{-\tau-1/2}) \\
&\qquad\qquad \le \frac{\sqrt{2\log\log n}}{\sqrt{n}} \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{n}} \frac{\sqrt{\log\log K_n}}{\sqrt{k_n}} + O_p(\log n/n) + O_p(n^{-\tau-1/2})
\end{aligned}
$$

where the last two inequalities result from the continuity module of Gaussian processes and the fact that $g(b) \ge \underline{b} > 0$ for all b (GPV Proposition 1). Recall that $k_n \propto \frac{n}{\log n}$, we con conclude that the right hand side is of order $O_p((n/\log n)^{-2/3})$.

Lemma 8. *Suppose Assumptions 3 and 4 are satisfied, the* $\|\widehat{V} - V_n\| = O_p((n/\log n)^{-2/3})$.

Proof. The conclusion holds by Lemmas 5 to 7. □

Lemma 9. *Let* $z_{(i)} = n(b_{(i)} - b_{(i-1)})$ *and* $w_i = ((i-1)/n - \alpha)\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u-\alpha)du$. *Suppose Assumption 3 is satisfied, then* $\sum_i z_{(i)}w_i = o_p(1/\sqrt{nh})$.

Proof. Since b_i has bounded support, it is without loss of generality to prove the case when b_i follows the uniform distribution. Pyke (1965, Section 2.1) shows that $z_{(i)}$ are identically distributed across i. Furthermore, $\mathbb{E}[z_{(i)}] = n(n+1)^{-1}$, $V(z_{(i)}) = n^3(n+1)^{-2}(n+2)^{-1}$ and $\text{Cov}(z_{(i)}z_{(j)}) = -n^2(n+1)^{-2}(n+2)^{-1}$. Let ρ_{ij} be the correlation coefficient, so $\rho_{ij} = 1$ if $i = j$, and $\rho_{ij} = -1/n$ otherwise.

Note first that $\mathbb{E}[\sum_i z_{(i)} w_i] = n(n+1)^{-1} \sum_i w_i = (1/h) \left(\int_0^1 (u-\alpha) K(u-\alpha/h) du + O(1/n) \right) = O(1/nh) = o_p(1/\sqrt{nh})$ since $\int uK(u) = 0$ by assumption. Next consider

$$V(\sum_i z_{(i)} w_i) = \sum_i w_i^2 V(z_{(i)}) + 2 \sum_{i \neq j} w_i w_j \mathrm{Cov}(z_{(i)}, z_{(j)}) = V(z_{(i)}) \left(\sum_i w_i^2 + 2 \sum_{i \neq j} w_i w_j \rho_{ij} \right).$$

Consider w_i, there exists a $u_i^* \in ((i-1)/n, i/n)$ such that

$$w_i = \left(\frac{i-1}{n} - \alpha \right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u-\alpha) du = \frac{1}{nh} \left(\frac{i-1}{n} - \alpha \right) K \left(\frac{u_i^* - \alpha}{h} \right),$$

Since the kernel function has bounded support, that is, $K(u) = 0$ if $|u| > \bar{K}$. Then $w_i \neq 0$ only if $|u_i^* - \alpha| \leq \bar{K}h$. Therefore the quantity $i - 1/n$ for nonzero w_i is around h neighborhood of α, which implies that each of the nonzero $|w_i|$ is of order $\frac{1}{nh} \times h = \frac{1}{n}$. Let i_α be the nearest integer to $n\alpha$, then we know $w_i \neq 0$ only if $|i - i_\alpha| \leq Cnh$ for some constant C, which implies that in the expression of $V(\sum_i z_{(i)} w_i)$, there are of order nh nonzero summands. Since each w_i is of order $1/n$, $\rho_{ij} = -1/n$ when $i \neq j$, $V(z_{(i)}) = O(1)$, the order of $V(\sum_i z_{(i)} w_i)$ is $O(nh \times (1/n)^2 + (nh)^2 \times (1/n)^3) = O(h/n)$, which is of smaller order than $1/nh$.

The above argument shows that $\mathbb{E}[\sum_i z_{(i)} w_i] = o_p(1/\sqrt{nh})$ and $V(\sum_i z_{(i)} w_i) = o_p(1/nh)$, therefore we can conclude that $\sum_i z_{(i)} w_i = o_p(1/\sqrt{nh})$.

Lemma 10. *For* $0 < \alpha < 1$, *let*

$$B_n(\alpha) \equiv \sum_{i=1}^{n} (i-1)(b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du - \frac{\alpha}{g(Q_b(\alpha))}.$$

If Assumptions 3 and 4 are satisfied, then $\sqrt{nh} B_n(\alpha) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$, *where constant* $\mathscr{B}$ *and* $\mathscr{V}$ *are defined below in the proof.*

Proof. Define $\tilde{B}_n(\alpha)$ as

$$\tilde{B}_n(\alpha) = \sum_{i=1}^{n} \alpha n (b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u)\, du - \frac{\alpha}{g(Q_b(\alpha))}$$

Note first when n is large,

$$
\begin{aligned}
n\sum_{i=1}^{n}(b_{(i)}-b_{(i-1)})\int_{\frac{i-1}{n}}^{\frac{i}{n}}K_h\left(\alpha-u\right)du &= n\sum_{i=1}^{n-1}b_{(i)}\int_{\frac{i-1}{n}}^{\frac{i}{n}}K_h\left(\alpha-u\right)du \\
-\,n\sum_{i=1}^{n-1}b_{(i)}\int_{\frac{i}{n}}^{\frac{i+1}{n}}K_h\left(\alpha-u\right)du &+ nb_{(n)}\int_{\frac{n-1}{n}}^{1}K_h\left(\alpha-u\right)du - nb_{(0)}\int_{0}^{1/n}K_h\left(\alpha-u\right)du \\
&\approx n\sum_{i=1}^{n-1}b_{(i)}\int_{\frac{i-1}{n}}^{\frac{i}{n}}K_h\left(\alpha-u\right)du - n\sum_{i=1}^{n-1}b_{(i)}\int_{\frac{i}{n}}^{\frac{i+1}{n}}K_h\left(\alpha-u\right)du.
\end{aligned}
$$

The last equality holds because under Assumption 4, when n is large, $K_h\left(t\right)=0$ for any $t\neq 0$. Recall that $K_h(\cdot)=(1/h)K(\cdot/h)$, we know that

$$
\begin{aligned}
\tilde{B}_n(\alpha) = \alpha n\sum_{i=1}^{n-1}b_{(i)}&\left\{\int_{\frac{i-1}{n}}^{\frac{i}{n}}K_h\left(\alpha-u\right)du-\int_{\frac{i}{n}}^{\frac{i+1}{n}}K_h\left(\alpha-u\right)du\right\} \\
&= \frac{\alpha}{h^2}\sum_{i=1}^{n-1}b_{(i)}\int_{\frac{i-1}{n}}^{\frac{i}{n}}K'\left(\frac{u-\alpha}{h}\right)du+O_p(1/n).
\end{aligned}
$$

By Welsh (1988, main theorem, part (ii)), under Assumptions 3 and 4 and $nh^5\to c$, $\sqrt{nh}(\tilde{B}_n(\alpha)-\frac{\alpha}{g(Q_b(\alpha))})\xrightarrow{d}N(\mathscr{B},\mathscr{V})$, where

$$
\mathscr{B}=-\frac{c^2\alpha}{6(I-1)}Q_b'''(\alpha)\int u^3K'(u)du \quad \mathscr{V}=\frac{\alpha^2}{c(I-1)^2}(Q_b'(\alpha))^2\int K^2(u)du.
$$

When $h=cn^{-r}$ for some $\frac{1}{5}<r<\frac{1}{2}$, $\sqrt{n^{1-r}}(\tilde{B}_n(\alpha)-Q_v(\alpha))\xrightarrow{d}N(0,\mathscr{V})$.

Lastly, let $z_{(i)}=n(b_{(i)}-b_{(i-1)})$ and $w_i=((i-1)/n-\alpha)\int_{\frac{i-1}{n}}^{\frac{i}{n}}K_h(u-\alpha)du$. Observe that $B_n-\tilde{B}_n=\sum_i z_{(i)}w_i$. Lemma 9 shows that it is of order $o_p(1/\sqrt{nh})$. Therefore we can conclude that $\sqrt{nh}B_n(\alpha)\xrightarrow{d}N(\mathscr{B},\mathscr{V})$. □

Appendix D. Compute the Greatest Convex Minorant of $V_n(\cdot)$

We now describe how to compute the greatest convex minorant of $V_n(\cdot)$.

First, consider the coordinate vectors of the piecewise linear function $V_n(\cdot)$: $\{(0,0),(1/n,V_n(1/n)),\ldots,(1,V_n(1))\}$. We find the smallest slope of each $(j/n,V_n(j/n))$ with respect to the origin, which defines the first partition on the g.c.m.. Let $j_1=\operatorname{argmin}_{j\in\{1,\ldots,n\}}\frac{V_n(j/n)-0}{j/n-0}$. The first partition is the line segment connecting $(0,0)$ and $(j_1,V_n(j_1/n))$.

Second, we find the next smallest slope, after removing the first partition from further consideration. In particular, consider the coordinate vectors $\{(j_1, V_n(j_1/n)), \ldots, (1, V_n(1))\}$. Let $j_2 = \operatorname{argmin}_{j \in \{j_1+1,\ldots,n\}} \frac{V_n(j/n) - V_n(j_1/n)}{j/n - j_1/n}$. The second partition is the line segment connecting $(j_1, V_n(j_1/n))$ and $(j_2, V_n(j_2/n))$.

We continue in this manner until we reach the end of the points $(1, V_n(1))$. The resulting coordinate vectors $\{(0,0), (j_1, V_n(j_1/n)), (j_2, V_n(j_2/n)), \ldots, (1, V_n(1))\}$ define the greatest convex minorant of $V_n(\cdot)$, which is also piecewise linear.